Math 152 (honors sections)

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**Reminder**

Second examination is Wednesday, November 3.

The exam covers through section 10.4.


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**Convergence tests so far**

- If $a_n \not\to 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

- A geometric series with $|\text{ratio}| < 1$ converges.

- Integral test for positive decreasing functions: the improper integral $\int_{1}^{\infty} f(x) \, dx$ and the corresponding series $\sum_{n=1}^{\infty} f(n)$ either both converge or both diverge.

- Inequality comparison test for positive terms: if $0 < a_n < b_n$ (at least for $n$ large) and if $\sum b_n$ converges, then $\sum a_n$ converges too.

- Limit comparison test for positive terms: if $\lim_{n \to \infty} \frac{a_n}{b_n}$ exists (finite limit), and if $\sum b_n$ converges, then $\sum a_n$ converges too.
Root test (not in book)

Example: \( \sum_{n=1}^{\infty} \frac{n}{2^n} \) converges

This is not a geometric series, and it is \textit{bigger} than the convergent geometric series \( \sum_{n=1}^{\infty} \frac{1}{2^n} \), so the comparison test does not seem to help.

Since \( \frac{n}{2^n} = \left( \frac{n^{1/n}}{2} \right)^n \), and since \( \lim_{n \to \infty} n^{1/n} = 1 \), we have \( \frac{n}{2^n} < \left( \frac{1.1}{2} \right)^n \) when \( n \) is large, so we can use the comparison test after all: compare to the convergent geometric series \( \sum_{n=1}^{\infty} \left( \frac{1.1}{2} \right)^n \).

Root test and ratio test

\textbf{Root test:} If \( 0 < a_n \), and if \( \lim_{n \to \infty} a_n^{1/n} < 1 \), then \( \sum_{n=1}^{\infty} a_n \) converges. Moreover, if \( \lim_{n \to \infty} a_n^{1/n} > 1 \), then \( \sum_{n=1}^{\infty} a_n \) diverges (because then \( a_n \not\to 0 \)). If \( \lim_{n \to \infty} a_n^{1/n} = 1 \), the test gives no information.

\textbf{Ratio test:} Exactly the same as the root test, except look at \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \) instead of \( \lim_{n \to \infty} a_n^{1/n} \).

Example: \( \sum_{n=1}^{\infty} \frac{3^n (n!)^2}{(2n)!} \).

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{3^{n+1}(n+1)!(n+1)!}{(2n+2)!} \cdot \frac{3^n (n!)^2}{(2n)!} = \lim_{n \to \infty} \frac{3(n+1)^2}{(2n+2)(2n+1)} = \frac{3}{4} < 1,
\]

so the original series converges.
Series with some negative terms

Negative terms can only help with convergence:
if \( \sum_{n=1}^{\infty} |a_n| \) converges, then so does \( \sum_{n=1}^{\infty} a_n \).

An absolutely convergent series converges.

Example: \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \) converges because \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges.

The two series sum to different values, however: \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12} \), and \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \).

Example: \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges (harmonic series),
yet \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) converges (in fact to the value \( \ln\left(\frac{1}{2}\right) \)).

Alternating series test

If \( a_n \downarrow 0 \), then \( \sum_{n=1}^{\infty} (-1)^n a_n \) converges.

Error estimate: When the alternating series test applies, the sum of the series is trapped between any two consecutive partial sums.

Example: \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \) converges,

\[
-1 + \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4} + \cdots < \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} < -1 + \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} < -0.94754 < \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} < -0.9459
\]

Exact value: \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} = \frac{7\pi^4}{720} \approx -0.947033 \).
Homework

- Read section 10.4, pages 605–610.
- Do the Suggested Homework problems for section 10.4.

Monday we will review for the exam and look at an old exam.