Math 409-502

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Problem 5 on the exam

Determine the radius of convergence of the power series $\sum_{n=0}^{\infty} \left( \frac{1 + 2^n}{1 + n^2} \right) x^n$.

Solution. Here is one of several valid methods.
First observation: since the open interval of convergence $(-R, R)$ is symmetric, we may as well assume that $x > 0$.
Second observation: now the asymptotic comparison test applies, so the new series $\sum_{n=1}^{\infty} \left( \frac{2^n}{n^2} \right) x^n$ converges for the same positive values of $x$.
By the root test, this new series converges when
$$1 > \lim_{q \to \infty} \frac{2x}{n^{2/q}} = 2x.$$
So the radius of convergence is $1/2$. 
Problem 4(b) on the exam

If a function \( g \) has a jump discontinuity at 0, and a function \( h \) is continuous at 0, then the product function \( gh \) has a jump discontinuity at 0.

True or false?

“Jump discontinuity” means that \( g \) has one-sided limits, but \( \lim_{x \to 0^-} g(x) \neq \lim_{x \to 0^+} g(x) \).

Since \( h \) is continuous, the product function \( gh \) has one-sided limits equal to \( h(0) \cdot \lim_{x \to 0^-} g(x) \) and \( h(0) \cdot \lim_{x \to 0^+} g(x) \). These one-side limits are equal when \( h(0) = 0 \) and unequal when \( h(0) \neq 0 \).

So the answer is “false”, but the statement is true most of the time (whenever \( h(0) \neq 0 \)).

Problem 4(a) on the exam

If a function \( f \) is locally bounded on an interval, then \( f \) is bounded on the interval.

True or false?

Theorem 10.4 on page 146 says the statement is true if the interval is compact.

On non-compact intervals, however, the statement is false. Example: \( 1/x \) on the open interval \((0,1)\).
Problem 3(b) on the exam

Prove from the \( \varepsilon \)-\( \delta \) definition that the function \( 1/x \) is continuous at the point 1.

Fix \( \varepsilon > 0 \). We must find \( \delta > 0 \) such that

\[
\left| \frac{1}{x} - 1 \right| < \varepsilon
\]

whenever \( |x - 1| < \delta \). Now

\[
\left| \frac{1}{x} - 1 \right| = \frac{|x - 1|}{|x|},
\]
and the difficulty is that the denominator could be small.

One way to handle the difficulty is to take \( \delta = \min\left(\frac{1}{2}, \frac{\varepsilon}{2}\right) \).

If \( |x - 1| < \delta \), then in particular \( |x - 1| < \frac{1}{2} \), so \( x > \frac{1}{2} \),

whence \( \frac{1}{x} < 2 \).

Then

\[
\frac{|x-1|}{|x|} \leq 2 |x - 1| < 2\delta \leq \varepsilon.
\]

Thus we have the required \( \delta \).

Homework

Use the \( \varepsilon \)-\( \delta \) definition of continuity to prove that

1. the function \( 1/x^2 \) is continuous at the point 1;
2. the function \( 1/x \) is continuous at the point \( 1/10 \).