Exercise on the definition of infinite products

The goal of this exercise is to arrive at a reasonable definition of what it means for an infinite product to converge. I am not telling you a definition, because there is no universally accepted one.

1. Based on your experience with infinite sums, formulate what you think is a reasonable definition of the statement

\[ \prod_{n=0}^{\infty} a_n = L, \]

where the symbol \( \prod \) stands for product, and the \( a_n \) and \( L \) are complex numbers.

2. What does your definition say about the following two examples?

(a) \[ \prod_{n=0}^{\infty} \left( \frac{1}{2^n} \right) \]  
(b) \[ \prod_{n=0}^{\infty} (1 - 2^n) \]

A necessary condition for convergence of an infinite sum \( \sum b_n \) is that \( b_n \rightarrow 0 \). The analogous statement for infinite products would be that a necessary condition for convergence of \( \prod a_n \) is that \( a_n \rightarrow 1 \).

3. Is this a reasonable condition? Is it in accord with your definition of convergence of infinite products?

Since the logarithm of a product is the sum of the logarithms, it is sometimes said that \( \prod a_n \) converges if and only if \( \sum \log a_n \) converges.

4. Is this a reasonable condition? Is it in accord with your definition of convergence of infinite products?

5. Finally, reformulate your definition of convergence of infinite products, taking account of the issues developed in parts 2–4.