Exercise on the Weierstrass $\wp$ function

This exercise provides a direct solution (without using series expansions) of the homework problem stating that

$$\det \begin{vmatrix} \wp(z_1) & \wp'(z_1) & 1 \\ \wp(z_2) & \wp'(z_2) & 1 \\ \wp(z_1 + z_2) - \wp'(z_1 + z_2) & 1 \end{vmatrix} = 0,$$  

(*)

where $\wp$ is the Weierstrass doubly periodic meromorphic function defined with respect to an arbitrary lattice, and all the terms in the determinant are assumed to be finite (that is, the points $z_1$, $z_2$, and $z_1 + z_2$ are not points of the lattice).

1. Deduce from the residue theorem that if $f$ is meromorphic in a simply connected region, if $\gamma$ is a simple closed curve in the region, if the zeroes of $f$ inside $\gamma$ are $a_1, \ldots, a_m$ (repeated according to multiplicity), if the poles of $f$ inside $\gamma$ are $b_1, \ldots, b_n$ (repeated according to multiplicity), and if $f$ has no zeroes or poles on the curve $\gamma$, then

$$\frac{1}{2\pi i} \int_\gamma \frac{zf''(z)}{f(z)} \, dz = \sum_{j=1}^{m} a_j - \sum_{k=1}^{n} b_k.$$  

(Compare Exercise 2 on page 174, which you did last semester.)

2. Suppose $f$ is a doubly periodic meromorphic function that has a multiple pole at the origin and no other poles within a fundamental parallelogram centered at the origin. Deduce from part 1 that if the zeroes of $f$ in the parallelogram are $a_1, \ldots, a_m$, then $a_1 + \cdots + a_m = 0$.\footnote{Correction added 19 April: the conclusion should be that $a_1 + \cdots + a_m \equiv 0$ modulo the lattice; that is, $a_1 + \cdots + a_m$ is an element of the lattice.}

3. Fix a lattice and a corresponding Weierstrass $\wp$ function. Let $z_1$ and $z_2$ be distinct complex numbers such that $z_1$, $z_2$, and $z_1 + z_2$ are not points of the lattice, and consider the determinant function $D$ defined by

$$D(z) = \det \begin{vmatrix} \wp(z_1) & \wp'(z_1) & 1 \\ \wp(z_2) & \wp'(z_2) & 1 \\ \wp(z) & \wp'(z) & 1 \end{vmatrix}.$$  

Show that $D$ is a doubly periodic meromorphic function of $z$ of order 3 that has zeroes at $z_1$ and $z_2$. Use part 2 to locate a third zero of $D$, and deduce equation (*) from the symmetry properties of $\wp$ and $\wp'$.\footnote{Correction added 19 April: the conclusion should be that $a_1 + \cdots + a_m \equiv 0$ modulo the lattice; that is, $a_1 + \cdots + a_m$ is an element of the lattice.}