Exercise on univalent functions

A holomorphic function that takes no value more than once is called variously one-to-one or univalent or *schlicht*. The latter term is a German word that has no exact English equivalent. (There is a story that a student once asked W. F. Osgood if there were an English term for schlicht functions. Osgood is supposed to have replied, “You *could* call them univalent functions, and everyone would know that you meant schlicht.”)

Local univalence

1. Show that if \( f \) is a holomorphic function, and \( a \) is a point at which the derivative \( f'(a) \neq 0 \), then there is a neighborhood of \( a \) in which \( f \) is one-to-one.\(^1\)

2. Give an example of an entire function that is locally one-to-one but not globally one-to-one.\(^2\)

Global univalence

One needs either a formula or some global information to show that a holomorphic function is univalent in the large. Let’s consider functions whose domain is the unit disc.

3. Show that the *Koebe function* \( z \mapsto \frac{z}{(1-z)^2} \) maps the (open) unit disc one-to-one onto the plane minus a slit from \(-1/4\) to \(-\infty\) along the negative real axis.\(^3\)

4. Suppose that \( f \) is holomorphic on a neighborhood of the closure of the unit disc, and \( f \) is one-to-one on the boundary of the disc. Show that \( f \) is one-to-one on the closed disc, and the image of the open disc is the region inside the image of the boundary.\(^4\)

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\(^1\)Hint: \( \frac{\pi}{\log \frac{1}{\rho}} \) see Theorem 5.2.2

\(^2\)Hint: Consider \( e^z \) whose derivative is nowhere zero.

\(^3\)Hint: \( 1 - \frac{1-z}{z} = \frac{z}{1+z} \)

\(^4\)This is problem 17 on page 178. Hint: \( \frac{\sin \pi \rho}{\rho} \)