1. \(x-2x^2\) and \(1+x+x^2\) are not multiple of each other so they are linearly independent. I keep both

\[1+2x-x^2 = (x-2x^2) + (1+x+x^2)\]

so I can discard \(1+2x-x^2\).

\[1+3x^2 = (1+x+x^2) - (x-2x^2)\]

so I can discard \(1+3x^2\).

Consider now the remaining 3 vectors; \(1+x+x^2, x-2x^2, 1+x\)

If \(c_1(1+x+x^2) + c_2(x-2x^2) + c_3(1+x) = 0\)

we get the following system for \(c_1, c_2, c_3\):

\[
\begin{pmatrix}
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 \\
1 & -2 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
c_3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
-2c_1 + c_2 = 0 \\
-c_1 + c_3 = 0 \\
2c_2 + c_3 = 0
\end{pmatrix}
\]

So the vectors \(1+x+x^2, x-2x^2, 1+x\) are independent and since \(\dim P_3 = 3\) they form a basis.

2) a) Let \(A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\), \(A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\), \(A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\)

Then \(A_1, A_2, A_3\) are lin independent

If \(c_1 A_1 + c_2 A_2 + c_3 A_3 = 0 \Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0 \Rightarrow c_1 = c_2 = c_3 = 0\)

2) \(A_1, A_2, A_3\) span the space of symmetric matrices.

Any symmetric \(2 \times 2\) matrix is of the form

\[
\begin{pmatrix}
a & b \\
b & c
\end{pmatrix} = a A_1 + b A_2 + c A_3,
\]

so \(A_1, A_2, A_3\) form a basis for the space of \(2 \times 2\) symmetric matrices. The dimension of this space is 3.
6) For the space of \( n \times n \) symmetric matrices, we form a basis by first taking all of those matrices who have one 1 in the diagonal and 0 everywhere else. Then we add those matrices who have 2's in symmetric position about the diagonal and 0 everywhere else. These together form a basis. The number of those of the first kind is \( \# \text{entris in diagonal} = n \)

The number of those of the second kind is the number of entries above the diagonal = \( \frac{n^2-n}{2} \)

The dimension is \( \frac{n^2-n}{2} + n = \frac{n^2+n}{2} = \frac{n(n+1)}{2} \).

3, a) \( W = \begin{vmatrix} \sin x & \cos x & \sin (x + \frac{\pi}{3}) \\ \cos x & -\sin x & \cos (x + \frac{\pi}{3}) \\ -\sin x & -\cos x & -\sin (x + \frac{\pi}{3}) \end{vmatrix} = 0 \) since the determinant has a row of zeros.

Because the determinant is identically 0 we cannot include anything about the linear dependence of the functions from the Wronskian. We know however from trigonometry that

\[ \sin(x + \frac{\pi}{3}) = \sin x \cos \frac{\pi}{3} + \sin \frac{\pi}{3} \cos x \]

So:

\[ (\cos \frac{\pi}{3}) \sin x + (\sin \frac{\pi}{3}) \cos x + (-1) \sin(x + \frac{\pi}{3}) = 0 \]

The functions are linearly dependent.

6) To be able to compute the Wronskian \( q_1, q_2 \) should be in \( C^1[-2,2] \), that is have a continuous derivative in \([-2,2]\) but \( q_2 = \frac{1}{3} x^{-\frac{2}{3}} \) does not exist at \( x=0 \) so \( q_2 \notin C^1[-2,2] \).
c) Suppose \( e_1, q_1 + e_2 q_2 = 0 \)
\( c_1 x + c_2 x^{\frac{1}{3}} = 0 \)

At \( x = 1 \):
\[
\begin{align*}
    c_1 + c_2 &= 0 \\
    -2R_1 + R_2 &\rightarrow R_2
\end{align*}
\]

At \( x = \frac{1}{2} \):
\[
\begin{align*}
    c_1 + c_2 &= 0 \\
    c_1, 2c_2, 2c_3 &= 0
\end{align*}
\]

\( c_1 + c_2 = 0 \)
\[
(2^{\frac{1}{3}} - 2) c_2 = 0 \quad \Rightarrow \quad c_2 = 0
\]

\( c_1 + c_2 = 0 \quad \Rightarrow \quad c_1 = 0
\]

The functions \( c_1, c_2 \) are linearly independent.

4. Suppose \( [\overline{u_1}, \overline{u_2}] \) and \( [\overline{v_1}, \overline{v_2}] \) be bases for \( U \) and \( V \) respectively. Suppose \( U \cap V = \{0\} \).

If \( c_1 \overline{v_1} + c_2 \overline{v_2} + c_3 \overline{u_1} + c_4 \overline{u_2} = 0 \)

then \( c_1 \overline{v_1} + c_2 \overline{v_2} = -c_3 \overline{u_1} - c_4 \overline{u_2} \)

so \( c_1 \overline{v_1} + c_2 \overline{v_2} \in V \) and \(-c_3 \overline{u_1} - c_4 \overline{u_2} \in U \)

so \( c_1 \overline{v_1} + c_2 \overline{v_2} = -c_3 \overline{u_1} - c_4 \overline{u_2} = 0 \)

since \( U \cap V = \{0\} \), \( \overline{v_1}, \overline{v_2} \) are lin indp \( c_1 = c_2 = 0 \)

Since \( \overline{v_1}, \overline{v_2} \) are lin indp \( c_3 = c_4 = 0 \)

Since \( \overline{u_1}, \overline{u_2} \) are lin indp \( \overline{v_1}, \overline{v_2} \) are linearly independent.

This shows that \( \overline{u_1}, \overline{u_2}, \overline{v_1}, \overline{v_2} \) are linearly independent. This cannot happen since in a 3-dim space, any 4 vectors are linearly dependent. So \( U \cap V \neq \{0\} \).

5. First let us find the RREF of \( A \).

\[
A = \begin{pmatrix}
1 & 2 & -1 & 3 & 2 \\
-1 & 3 & 0 & 1 & 1 \\
1 & 2 & 1 & 4 & 2 \\
2 & -1 & 1 & 3 & 1
\end{pmatrix}
\]

\( R_1 + R_2 \rightarrow R_2 \)

\( -R_1 + R_3 \rightarrow R_3 \)

\( -2R_1 + R_4 \rightarrow R_4 \)

\( R_2 + R_4 \rightarrow R_4 \)

\( R_3 + R_4 \rightarrow R_4 \)

\[
\begin{pmatrix}
1 & 2 & -1 & 3 & 2 \\
0 & 5 & -1 & 4 & 3 \\
0 & 0 & 2 & 1 & 0 \\
0 & 0 & 2 & 1 & 0
\end{pmatrix}
\]
\[
\left(1 2 -1 3 \begin{array}{c} 2 \\ 0 5 -1 4 \\ 0 2 1 0 \\ 0 0 0 0 \end{array} \right) \frac{1}{5} R_2 \rightarrow R_2 \\
\left(0 1 -\frac{4}{3} \frac{3}{5} \frac{8}{5} \right) \frac{1}{2} R_3 \rightarrow R_3 \\
\left(0 0 1 \frac{1}{2} 0 \right) \frac{1}{5} R_3 + R_2 \rightarrow R_2
\]

\[
\left(1 2 0 \frac{7}{2} 1 \begin{array}{c} 2 \\ 0 1 0 \frac{9}{10} \frac{3}{5} \\ 0 0 1 \frac{1}{2} 0 \\ 0 0 0 0 \end{array} \right) -2 R_2 + R_1 \rightarrow R_1
\]

\[
U = \left(1 0 0 \frac{17}{10} \frac{4}{5} \begin{array}{c} \\ 0 1 0 \frac{9}{10} \frac{3}{5} \frac{3}{5} \\ 0 0 1 \frac{1}{2} 0 \\ 0 0 0 0 \end{array} \right) \Rightarrow \text{RREF}
\]

So \( (0 1 0 \frac{9}{10} \frac{3}{5} ) \), \( (0 1 0 \frac{9}{10} \frac{3}{5} ) \), \( (0 0 1 \frac{1}{2} 0 ) \) is a basis for Row space \( \text{R}(A) \) and \( \text{rank}(A) = \dim \text{row space}(A) = 3 \)

Row 1 \( (A) = 1(1, 0, 0, \frac{17}{10}, \frac{4}{5}) + 2(0, 0, 0, \frac{3}{5}, 0) - 1(0, 0, 0, 1, 0) = (1, 2, -1, 3, 2) \)

Row 2 \( (A) = -1(1, 0, 0, \frac{17}{10}, \frac{4}{5}) + 3(0, 0, 0, \frac{3}{5}, 0) + 0(0, 0, 0, 1, 0) = \)

\[
\left( -1, 3, 0, 1, 1 \right)
\]

Row 3 \( (A) = 1(1, 0, 0, \frac{17}{10}, \frac{4}{5}) + 2(0, 1, 0, \frac{9}{10}, \frac{2}{5}) + 1(0, 0, 1, 0, \frac{1}{2}, 0) = \)

\[
\left( 1, 2, 1, 4, 2 \right)
\]

Row 4 \( (A) = 2(1, 0, 0, \frac{17}{10}, \frac{4}{5}) - 1(0, 0, 0, \frac{9}{10}, \frac{2}{5}) + 1(0, 0, 1, 0, \frac{1}{2}, 0) = \)

\[
\left( 2, -1, 1, 3, 1 \right)
\]

Since \( \bar{u}_1, \bar{u}_2, \bar{u}_3 \) are a basis for Row space \( \text{R}(U) \), \( \bar{a}_1, \bar{a}_2, \bar{a}_3 \)

are a basis for \( \text{null}(A) \).

Since \( \bar{u}_4 = \frac{17}{10} \bar{u}_1 + \frac{9}{10} \bar{u}_2 + \frac{1}{2} \bar{u}_3 \Rightarrow \bar{a}_4 = \frac{17}{10} \bar{a}_1 + \frac{9}{10} \bar{a}_2 + \frac{1}{2} \bar{a}_3 \)

Since \( \bar{u}_5 = \frac{4}{5} \bar{u}_1 + \frac{3}{5} \bar{u}_2 \Rightarrow \bar{a}_5 = \frac{4}{5} \bar{a}_1 + \frac{3}{5} \bar{a}_2 \)

b) Since \( \bar{a}_1, \bar{a}_2, \bar{a}_3 \) are a basis for \( \text{null}(A) \), we have 2 free variables:

\[
\left( x_1, x_2, x_3, x_4, x_5 \right) = \left( s, t, \frac{17}{10} s - \frac{9}{10} s - \frac{4}{5} t, -\frac{1}{2} s, t \right)
\]

Then

\[
\begin{align*}
X_1 &= \frac{17}{10} s - \frac{4}{5} t \\
X_2 &= -\frac{9}{10} s - \frac{3}{5} t \\
X_3 &= -\frac{1}{2} s \\
X_4 &= s \\
X_5 &= t
\end{align*}
\]

The vectors \( (-\frac{17}{10}, -\frac{9}{10}, -\frac{1}{2}, 1, 0) \) and \( (-\frac{4}{5}, -\frac{3}{5}, 0, 0, 1) \) form a basis for \( \text{null}(A) \).

So \( \text{nullity}(A) = \dim \text{null}(A) = 2 \) and \( \text{rank}(A) + \text{nullity}(A) = 3 + 2 = 5 \)

Verifying the theorem.