1. Recall, the trace of a matrix \( A \) denoted \( \text{tr}(A) \) is the sum of its diagonal elements.

a) Show that if \( A, B \) are \( 2 \times 2 \) matrices \( \text{tr}(AB) = \text{tr}(BA) \). (2 pts)

b) Bonus show that if \( A, B \) are \( n \times n \) matrices then \( \text{tr}(AB) = \text{tr}(BA) \). (2 pts)

c) Show that if \( A, B \) are similar \( \text{tr}(A) = \text{tr}(B) \). (2 pts)

d) Let \( A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \). Which of the following matrices is (3 pts)
similar to \( A \)? Justify all your answers: If the answer is no, explain why.
\( \text{S}^{-1} \text{M} \text{S} = A \)
\( \text{S}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \)
\( \text{B} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \)
\( \text{C} = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \)
\( \text{D} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \)

2. a) Let \( \bar{x}, \bar{y} \) be vectors in \( \mathbb{R}^2 \) and let \( \bar{p} = \begin{pmatrix} \bar{x}^T \\ \bar{y}^T \end{pmatrix} \) and \( \bar{z} = \bar{x} - \bar{p} \). Show that \( \bar{p} \perp \bar{z} \). (2 pts)

b) Let \( \bar{x} = (1, 1, 1)^T \), \( \bar{y} = (1, 1, -1)^T \). Find vectors \( \bar{a} \) such that \( \bar{x} = \bar{a} + b \bar{y} \) and \( \bar{a} \perp \bar{y} \). (2 pts)

c) Verify Pythagoras theorem \( \| \bar{x} \|^2 = \| \bar{a} \|^2 + \| b \|^2 \). (1 pt)

3. Let \( P_3 \) be the space of polynomials of degree \( \leq 3 \) equipped with the inner product \( \langle p, q \rangle = \int_0^1 p(x)q(x)dx \).

Find a polynomial \( p = a + bx + cx^2 \in P_3 \) such that \( p \perp x, p \perp x^2 \in P_3 \) and \( \| p \| = 1 \). (3 pts)

4. a) Show that the formula
\[ \langle (x_1, x_2)^T, (y_1, y_2)^T \rangle = 3x_1y_1 + 2(x_1y_2 + x_2y_1) + 2x_2y_2 \]
defines an inner product on \( \mathbb{R}^2 \).

b) With the inner product defined in a) compute \( \| (1, -2)^T \| \) and find a real number \( a \) such that \( (1, -2)^T \perp (a, 1)^T \). (2 pts)

c) Does the formula
\[ \langle (x_1, x_2)^T, (y_1, y_2)^T \rangle = 3x_1y_1 + 2(x_1y_2 + x_2y_1) + x_2y_2 \]
define an inner product on \( \mathbb{R}^2 \)? Justify your answer.