1. Let \( \{a_n\} \) be the sequence defined recursively by:
\[
a_0 = \frac{1}{2}, \quad a_1 = 1, \quad a_2 = 2, \quad a_n = a_{n-1} + a_{n-2} + 2a_{n-3} \quad \text{for } n \geq 3.
\]
Compute \( a_4, a_5, a_6, \ldots \). Guess a formula for \( a_n \).
Prove your guess using induction. \( (5 \text{ pb}) \)

2. Find all pairs \( x, y \) of positive integers such that \( \gcd(x, y) = 30 \) and \( \text{lcm}(x, y) = 180 \). \( (5 \text{ pb}) \)

In problems 3, 4, 5 \( \{f_n\} \) denotes the Fibonacci sequence defined by:
\[
f_0 = 0, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2} \quad \text{for } n \geq 2.
\]

3. Prove by induction that \( f_n > \alpha^{n-2} \quad \text{for all } n \geq 3 \).
(\( \alpha \) is the positive root of \( x^2 = x + 1 \), \( \alpha = \frac{1 + \sqrt{5}}{2} \)). \( (5 \text{ pb}) \)

4. Prove that \( f_n \) is even if and only if \( 3 \mid n \). \( (5 \text{ pb}) \)

5. Prove by induction:
\[
\sum_{j=0}^{n} \binom{n-j}{j} = f_{n+1}.
\]

Here \( \lfloor x \rfloor \) denotes the largest integer not exceeding \( x \)
and \( \binom{n}{k} \) is the binomial coefficient. \( (5 \text{ pb}) \)