MATH 603 Homework Set VI
Due Friday, 5 December 2014

1. A vector field $\mathbf{F}(\mathbf{x})$ on $\mathbb{R}^n$ is called a Central Field provided it has the form $\mathbf{F}(\mathbf{x}) = \phi(\mathbf{x}) \mathbf{x}$ where $\phi(\mathbf{x})$ is a scalar valued function.

(a) Assume $\phi(\mathbf{x}) = \alpha(r)$ where $r = |\mathbf{x}|$, i.e. $\phi(\mathbf{x})$ is constant on spheres centered at the origin. Show that $\mathbf{F}(\mathbf{x})$ is conservative.

(b) Prove the converse of part (a), i.e. If a central vector field $\mathbf{F}(\mathbf{x}) = \phi(\mathbf{x}) \mathbf{x}$ is conservative, then $\phi(\mathbf{x}) = \alpha(r)$ for some scalar valued function $\alpha(r)$.

(c) Find all conservative central fields in $\mathbb{R}^n$ that satisfy $\text{Div}(\mathbf{F}) = 0$.

2. This problem is expressed using Polar Coordinates in $\mathbb{R}^2$. Consider a vector field of the form $\mathbf{F}(r, \theta) = \phi(r, \theta) j_2(\theta)$, where $j_1(\theta) = \cos(\theta)i_1 + \sin(\theta)i_2$ and $j_2(\theta) = -\sin(\theta)i_1 + \cos(\theta)i_2$.

(a) Show that $\text{Curl}(\mathbf{F})(r, \theta) = 0$ for $r > 0$ if and only if $\phi(r, \theta) = \psi(\theta)/r$ for some scalar valued function $\psi(\theta)$.

(b) If in part (a), $\int_0^{2\pi} \psi(\theta) d\theta \neq 0$, show that the vector field $\mathbf{F}$ is not conservative. Explain why this does not contradict the theorem proved in class that says “Curl free” implies conservative.

3. Consider the mapping $\mathbf{f}(\cdot) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given in Spherical Coordinates as $(R, \Theta, \Phi) \rightarrow (\rho(R, \Phi), \theta(\Theta, \Phi), \phi(\Phi))$ with $\rho(R, \Phi) = R f(\Phi)$, $\theta(\Theta, \Phi) = \Theta + \omega \Phi$, $\phi(\Phi) = g(\Phi)$, $\omega$ being a given constant. Geometrically, this mapping takes a cone with apex at the origin and central axis coincident with the positive $Z$-axis into another such cone with different opening angle and with a torsional twist. Using the notation from problem (2), one has $\mathbf{f}(R, \Theta, \Phi) = \rho(R, \Phi)k_1(\theta(\Theta, \Phi), \phi(\Phi))$. Let $F$ denote the second order tensor $\nabla \mathbf{f}$.

(a) Show that $F = a_{mn}(\Phi) k_m(\theta(\Theta, \Phi), \phi(\Phi)) \otimes k_n(\Theta, \Phi)$ with $a^{11} = f(\Phi)$, $a_{13} = f'(\Phi)$, $a^{22} = f(\Phi) \sin(g(\Phi))/\sin(\Phi)$, $a^{23} = \omega f(\Phi) \sin(g(\Phi))$, $a^{33} = f(\Phi)g'(\Phi)$ with all other $a^{kl}$ equal zero. ($f'(\Phi)$ denotes differentiation with respect to $\Phi$.)

(b) Show that $\det(F) = f(\Phi)^3 g'(\Phi) \sin(g(\Phi))/\sin(\Phi)$. 

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