Section 7.1 Probability Terminology

- A **sample space** is the set that contains all possible results (outcomes) of an experiment. Basically it is the Universal set.
- A **sample point** is an outcome of an experiment.
- An **event** is a subset of the sample space.
- A **simple event** is an event that contains only one element.
- Mutually exclusive means the same as disjoint.
- All of the set operations (intersection, union, and compliment) work the same with events.

Sections 7.2, 7.3, and 7.4 Probability and its rules

- **Probability** is a number assigned to each outcome of an experiment which gives that outcomes chance of occurring relative to the other outcomes.
- Rules for probability
  - The probability of a single outcome is between 0 and 1 inclusive.
  - The sum of the probability of all outcomes is 1.
- There are three methods to assign probability.
  - B.S. method: Just make up a number. (Not an advisable method for test questions.)
  - Empirical Method: Assigning probability based off of data collected.
  - Theoretical Method: Using counting to assign probability.
- The basic idea of assigning probability (this is taken with a grain or two of salt) is number of ways that and outcome can happen divided by the total number of trials.
- A sample space is called **uniform** or **equally likely** if every outcome has the same probability.
- Probability of events.
  - The probability of the Empty set is 0, i.e. \( P(\emptyset) = 0 \)
  - The probability of the Universal set is 1, i.e. \( P(U) = 1 \)
  - If \( E \subseteq S \), then \( P(E) \) is the sum of the probability of all the outcomes in the event \( E \).
  - If \( E \subseteq S \) and \( S \) is a uniform sample space, then \( P(E) = \frac{n(E)}{n(S)} \).
- These two formulas should also be known.
  \[
P(A) + P(A^C) = 1
\]
  \[
P(A \cup B) = P(A) + P(B) - P(A \cap B)
\]
  If \( A \) and \( B \) are mutually exclusive then \( P(A \cap B) = 0 \). Thus the above formula reduces to \( P(A \cup B) = P(A) + P(B) \)
• Probability can also be computed using Tables and Venn Diagrams.

i.e. This table classifies the English, History, Math, and Poly Sci majors at State U according to their year. (There are no double majors.)

<table>
<thead>
<tr>
<th></th>
<th>Freshmen(F)</th>
<th>Sophomores(Soph)</th>
<th>Juniors(J)</th>
<th>Seniors(Sr)</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>English(E)</td>
<td>64</td>
<td>35</td>
<td>31</td>
<td>41</td>
<td>171</td>
</tr>
<tr>
<td>History(H)</td>
<td>55</td>
<td>41</td>
<td>33</td>
<td>52</td>
<td>181</td>
</tr>
<tr>
<td>Math(M)</td>
<td>29</td>
<td>32</td>
<td>50</td>
<td>69</td>
<td>180</td>
</tr>
<tr>
<td>Poly Sci(PS)</td>
<td>70</td>
<td>33</td>
<td>41</td>
<td>37</td>
<td>181</td>
</tr>
<tr>
<td>Totals</td>
<td>218</td>
<td>141</td>
<td>155</td>
<td>199</td>
<td>713</td>
</tr>
</tbody>
</table>

If a student is selected at random, find the probability that

1. The student is a History major and a Sophomore.
2. The student is not a Sophomore and is an English major.
3. The student is a Math major or is a Senior.

**Section 7.5 and 7.6 Conditional Probability.**

• Probability can change when extra information is provided. This type of probability is referred to as **conditional probability**. The reason that the probability changes is that the extra information reduces the sample space since outcomes that can not occur are “removed” from the sample space. i.e. You are taking a multiple choice test with each question having 5 answers. If you randomly guess at a question, then you would have a probability of \(\frac{1}{5}\) or .20 of getting the question correct. If you realize that the question is about chemical elements and the last choice is Donald Duck, then you can eliminate that choice. Now there are only 4 answers to guess from so you have a probability of \(\frac{1}{4}\) or .25 of getting the question correct. This extra knowledge has increased your chance of getting the question correct.

• Notation: \(P(A|B)\) is read as what is the probability of event A occurring given (knowing) that event B has occurred. The second part is always the given part.

• Conditional probability is easy if you have a table or a chart. Using the chart at the top of this page. If a student is selected at random

  – Compute the probability that the student is a Senior knowing that the student is a math major. i.e. \(P(Sr|M)\). We are given that the student is a math major. This knowledge reduces the above table to a single row.

    \[
    \begin{array}{c|c|c|c|c}
    \text{Math(M)} & \text{Freshmen(F)} & \text{Sophomores(Soph)} & \text{Juniors(J)} & \text{Seniors(Sr)} \\
    \hline
    29 & 32 & 50 & 69 & 180 \\
    \end{array}
    \]

    Solution: \(P(Sr|M) = \frac{69}{180}\)

  – Compute the probability that the freshman selected is an English major. i.e. \(P(E|F)\).

    \[
    \begin{array}{c|c}
    \text{Freshmen(F)} & \text{English(E)} \\
    \hline
    64 & \text{History(H)} \\
    55 & \text{Math(M)} \\
    29 & \text{Poly Sci(PS)} \\
    70 & \text{Totals} \\
    \hline
    218 & 
    \end{array}
    \]

    We are given that the student is a freshman. This knowledge reduces the above table to only the Freshmen column (see right).

    Thus the solution is \(P(E|F) = \frac{64}{218}\).
Note: The technique of reducing a table to the given row or column only works because we are just as likely to pick one student as any another student. If the sample space is not equally likely or drawing a chart is not possible, then the formula that converts conditional probability to regular probability is very useful. This is a must know formula. \[ P(A|B) = \frac{P(A \cap B)}{P(B)} \]

Tree diagrams are also used with probability. Let's take a small example which has only two “actions”. Each action is represented as a level on the tree and the results of that action are the branches for that level. For our example, let \( J \) and \( J^C \) be the results of the first “action” and \( K \) and \( K^C \) be the results of the second “action”. The tree representing this example is on the right. Included on this tree is the formal probability notation for each section of the tree. Notice the first level of the tree is regular probability and the any level after that is conditional probability.

If we are given this tree and want to know the values of \( P(K|J) \), \( P(J \cap K) \), \( P(K) \) and \( P(J|K) \).

- **\( P(K|J) \):** just read the value from the section of the tree marked \( P(K|J) \).
- **\( P(J \cap K) \):** The formula, \( P(K|J) = \frac{P(K \cap J)}{P(J)} \), can be manipulated to \( P(J) \times P(K|J) = P(K \cap J) \). Or \( P(J \cap K) = P(J) \times P(K|J) \) since \( J \cap K = K \cap J \). Now read the values from the sections of the tree labeled \( P(J) \) and \( P(K|J) \) and multiply them together.
- **\( P(K) \):** \( K \) is a result for the second level of the tree. This means that \( K \) happens sometimes with \( J \) and sometimes with \( J^C \). Because of this we have to break \( P(K) \) into two parts: \( P(K) = P(K \cap J) + P(K \cap J^C) \). The work for computing \( P(J \cap K) \) lets us change the formula to \( P(K) = P(J) \times P(K|J) + P(J^C) \times P(K|J^C) \). Now read off the values located in the needed parts of the tree and plug them into the formula.
- **\( P(J|K) \):** This formula is sometimes referred to backwards probability since we are given a result for the second level and are asked about a result for the first level. This backwards probability formula is in section 7.6 of the book and is called Bayes’ Theorem. The actual formula for Bayes’ Theorem can be difficult or messy to memorize. The easy way to do this type of question is to change the conditional probability to regular probability., i.e. \( P(J|K) = \frac{P(J \cap K)}{P(K)} \). The computations for \( P(J \cap K) \) and \( P(K) \) were just shown.

Normally when constructing trees, the only thing that we put in the trees is the label of the branch and the probability associated with that branch. The formal formulas referring to the sections of the tree are not really used in this class. You should know how to draw trees and use them to compute the required probabilities.
Interpret these Statements\(^1\)

Express these questions using the correct probability notation.

A factory makes widgets on three different machines; A, B and C; and widget is selected at random.

1. What is the probability that it is defective knowing that the widget was made on machine C?
2. If the widget was made on machine A, what is the probability that it is defective?
3. What is the probability that the widget selected from line B is defective?
4. What is the probability it is made on machine A and is not defective?
5. What is the probability the widget is selected from line B or is defective?
6. What is the probability that the defective widget was made on machine C?
7. If the widget was not defective, what is the probability that it came from machine B?
8. What is the probability that a widget is classified as defective and came from machine B?
9. What is the probability that the widget is defective and came from machine A or machine B?
10. What is the probability that the widget came from Machine B if the selected widget was not defective?

Section 7.5 Independence

- Two events, A and B, are said to be independent if \(P(A|B) = P(A)\) or \(P(B|A) = P(B)\). (Note: If one of these formulas is true, then so is the other one.) The first formula basically says that the event B occurring does not affect the probability that A occurs. The second formula says the same thing but in reverse. **VERY IMPORTANT NOTE:** A lot of students mix up independence with mutually exclusive. These two concepts are not even close.

- While these formulas are useful to define independence, we normally use the formula \(P(A \cap B) = P(A) \times P(B)\) when we want to show if two events are independent or when we want to use the concept of independence.

\(^1\)These will be answered in class