On the volume of the intersection of two $L^n$ balls

This note deals with the following problem, the case $p/q = 1/2$ of which was introduced to us by Vitali Milman: What is the volume left in the $L^n_p$ ball after removing a $t$-multiple of the $L^n_q$ ball? Recall that the $L^n_p$ ball is the set $\{t \in \mathbb{R}^n : \|t\|^p \leq 1\}$.

In Corollary 4 below we show that after normalizing Lebesgue measure so that the volume of the $L^n_p$ ball is one, the answer to the problem above is of order $e^{-c\frac{t}{p - n/p}}$, where $c$ and $T$ depend on $p$ and $q$ but not on $n$.

The main theorem in Section 3 deals with the corresponding question for the surface $S^n_{p,q}$.

The main theorem in Section 3 introduces a class of random variables to be used in the proof of the main theorem. These random variables are related to $L^n_p$ in the same way that Gaussian variables are related to $L^n_2$.

In Section 2 we introduce a class of random variables to be used in the proof of the main theorem. These random variables are related to $L^n_p$ in the same way that Gaussian variables are related to $L^n_2$.

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1. Introduction

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2. Preliminaries

Here we introduce a class of random variables to be used in the proof of the main theorem and summarize some of their properties. Fix a $0 < p < \infty$ and let $x, x_1, x_2, \ldots, x_n$ be independent random variables each with density function $c_p e^{-p} \Gamma t > 0$. Note that necessarily $c_p = p/\Gamma(1/p)$. The first claim is known though we could not locate a reference.

Lemma 1. Put $S = (\sum_{i=1}^n x_i^p)^{1/p}$, then $\left(\frac{x_1}{S}, \frac{x_2}{S}, \ldots, \frac{x_n}{S}\right)$ is uniformly distributed over the positive quadrant of the sphere of $l^n_p$, i.e., over the set

$\Delta_p = \{(t_1, t_2, \ldots, t_n) : t_i \geq 0, \sum t_i^p = 1\}$ equipped with the $(n-1)$-dimensional normalized Lebesgue measure. Moreover, $\left(\frac{x_1}{S}, \frac{x_2}{S}, \ldots, \frac{x_n}{S}\right)$ is independent of $S$.

Proof. For any Borel subset $A$ of $\Delta_p \Gamma$

$$P\left(\left(\frac{x_1}{S}, \frac{x_2}{S}, \ldots, \frac{x_n}{S}\right) \in A \middle| S = a\right) =$$

$$\lim_{\epsilon \to 0} \frac{P((x_1, \ldots, x_n) \in \mathbb{R}_+ A \& a - \epsilon \leq S \leq a + \epsilon)}{P(a - \epsilon \leq S \leq a + \epsilon)}$$

$$= \lim_{\epsilon \to 0} \int_{(a-\epsilon)^p < \sum t_i^p < (a+\epsilon)^p} e^{-\sum t_i^p} dt / \int_{(a-\epsilon)^p < \sum t_i^p < (a+\epsilon)^p} e^{-\sum t_i^p} dt$$

$$\leq \limsup_{\epsilon \to 0} e^{-(a-\epsilon)^p + (a-\epsilon)^p} \int_{(a-\epsilon)^p < \sum t_i^p < (a+\epsilon)^p} dt / \int_{(a-\epsilon)^p < \sum t_i^p < (a+\epsilon)^p} dt$$

$$= \lambda(A),$$

where $\lambda$ is the normalized Lebesgue measure on $\Delta$. Similarly

$$P\left(\left(\frac{x_1}{S}, \frac{x_2}{S}, \ldots, \frac{x_n}{S}\right) \in A \middle| S = a\right) \geq \lambda(A).$$

This proves that $P\left(\left(\frac{x_1}{S}, \frac{x_2}{S}, \ldots, \frac{x_n}{S}\right) \in A\right) = \lambda(A)$ and that $\left(\frac{x_1}{S}, \frac{x_2}{S}, \ldots, \frac{x_n}{S}\right)$ is independent of $S$. ■
In the next claim we gather some more properties of the random variables $x_i$.

**Lemma 2.** Let $x, x_1, \ldots, x_n$ be as above, then

1. $c_p$ is bounded away from zero and infinity when $p \to \infty$.

2. For all $h > 0$ and all $0 < p < \infty$, $E e^{-hx^p} = \left(\frac{1}{1+h}\right)^{1/p}$. In particular,

\[
E e^{-hx^p} \geq e^{-h/p} \quad \text{for all} \quad h > 0 \quad \text{and} \quad E e^{-hx^p} \leq e^{-h/2p} \quad \text{for all} \quad 0 < h \leq 1.
\]

3. For all $0 < u < \infty$ and all $0 < p < \infty$, $P(x^p > u) \geq \frac{c_p}{2p}e^{-2u}$. If $p \geq 1$ and $u \geq 1$, then also $P(x^p > u) \leq \frac{c_p}{2p}e^{-u/2}$. In particular, for $p \geq 1$ and all $u$, $P(x^p > u) \leq Ce^{-u/2}$ for some universal $C$.

4. For all $1 \leq p \leq q < \infty$, $E \left(\sum_{i=1}^{n} x_i^q\right)^{1/q}$ is equivalent, with universal constants, to $q^{1/p}n^{1/q}$, if $q \leq \log n$, and to $(\log n)^{1/p}$ otherwise.

**Proof.**

1. Follows easily from the fact that $c_p = p/\Gamma(1/p) = \Gamma(\frac{1}{p} + 1)^{-1}$.

2. is a simple computation.

3. is also simple. There is a sketch of the proof.

\[
P(x^p > u) = c_p \int_{u^{1/p}}^{\infty} e^{-t^p} \, dt
\]

\[
\geq c_p \int_{u^{1/p}}^{(u+1)^{1/p}} \frac{pt^{(p-1)}}{p(u + 1)^{(p-1)/p}} e^{-t^p} \, dt
\]

\[
= \frac{c_p}{p(u + 1)^{(p-1)/p}} \left(1 - \frac{1}{e}\right) e^{-u}
\]

\[
\geq \frac{c_p}{2p(u + 1)} e^{-u}
\]

\[
\geq \frac{c_p}{2p} e^{-2u}.
\]

The other inequality in 3 is proved in a similar way.

4. First note that for all $0 < p, q < \infty$

\[
E x^q = c_p \int_{0}^{\infty} t^q e^{-t^p} \, dt = \frac{c_p}{p} \Gamma\left(\frac{q + 1}{p}\right)
\]
so that by the triangle inequality and 1 if 1 \leq p \leq q < \infty

\[ E\left(\sum_{i=1}^{n} x_i^q\right)^{1/q} \leq \left(\sum_{i=1}^{n} E x_i^q\right)^{1/q} = \left(\frac{c_{p}}{p} \Gamma\left(\frac{q+1}{p}\right)\right)^{1/q} n^{1/q} \leq C q^{1/p} n^{1/q} \]

for some universal \( C \). For the lower bound in the case \( q \leq \log n \) divide \( \{1, 2, \ldots, n\} \) into approximately \( n/e^q \) disjoint sets of cardinality approximately \( e^q \) each then

\[
E\left(\sum_{i=1}^{n} x_i^q\right)^{1/q} = E\left(\sum_{j} \left(\sum_{i \in \sigma_j} x_i^q\right)^{q/q}\right)^{1/q} \\
\geq E\left(\sum_{j} \left(\max_{i \in \sigma_j} x_i\right)^q\right)^{1/q} \\
\geq \left(\sum_{j} \left(E_{\max_{i \in \sigma_j}} x_i\right)^q\right)^{1/q} \\
\geq c' \left(\log n\right)^{1/p} n^{1/q} \\
\geq c n^{1/p} n^{1/q}.
\]

Now for the case \( q > \log n \) we note first that by 3 if

\[
P(\max_{1 \leq i \leq n} x_i > t) \geq 1 - \left(1 - \frac{c_{p}}{2p} e^{-2t^p}\right)^n.
\]

For \( n \) smaller than an absolute multiple of \( p \) the lower bound follows easily from the fact that \( E x_1 \) is larger than a universal positive constant so assume that \( n \geq 20p/c_{p} \) and put \( t = 2^{-1/p} \left(\log \frac{n c_{p}}{2p}\right)^{1/p} \). Then for some universal \( C \)

\[
P(\max_{1 \leq i \leq n} x_i > c(\log n)^{1/p}) \geq 1/2.
\]

In particular \( E \max_{1 \leq i \leq n} x_i \geq c(\log n)^{1/p} \) which implies the lower bound in this case since \( \left(\sum_{i=1}^{n} x_i^q\right)^{1/q} \) is universally equivalent to \( \max_{1 \leq i \leq n} x_i \). The upper bound in this case though a bit harder is also standard and since we don’t use it in the sequel we shall leave it to the reader.
The statement in 4\textsuperscript{th} for the case $p = 2\Gamma$ was noticed by the first named author several years ago while seeking a precise estimate for the dimension of the Euclidean sections of $L^p_n$ spaces (see [MS] p.145 Remark 5.7). The original proof was more complicated. The proof presented here is an adaptation of a proof of the case $p = 2$ shown to us by J. Bourgain.

3. The main result

**Theorem 3.** For all $1 \leq p < q < \infty$ there are constants $c = c(p, q)$, $C = C(p, q)$ and $T = T(p, q)$ such that if $\mu$ denotes the normalized Lebesgue measure on the positive quadrant of the unit sphere of $L^p_n$ then

\[ \mu(\|u\|_{L^q_n} > t) \leq \exp(-ct^{p_n/p}) \]  
for all $t > T$, and

\[ \mu(\|u\|_{L^q_n} > t) \geq \exp(-Ct^{p_n/p}) \]  
for all $2 \leq t \leq \frac{1}{2} n^{\frac{1}{p} - \frac{1}{q}}$.

Moreover, for $q > 2p$ (or any other universal positive multiple of $p$), one can take $c(p, q) = \frac{\gamma}{p}$, $C(p, q) = \frac{\Gamma}{p}$ and $T(p, q) = \tau \min\{q, \log n\}^{1/p} \leq q^{1/p}$. Here $\gamma$, $\Gamma$ and $\tau$ are universal constants.

**Proof.** By Lemma 1 above\(\Gamma\)

\[ \mu(\|u\|_{L^q_n} > t) = P(n^{\frac{1}{p} - \frac{1}{q}}(\sum_{i=1}^{n} x_i^q)^{1/q} / (\sum_{i=1}^{n} x_i^p)^{1/p} > t) \]

where $x_i$ are independent random variables each with density $c_p e^{-x^p}$. Assume\(\Gamma\) for the simplicity of the presentation\(\Gamma\) that $n$ is even. Put $S = (\sum_{i=1}^{n} x_i^p)^{1/p}$ and let $p_j \Gamma \ j = 1, 2, \ldots, n/2$ be positive numbers with sum $\leq 1/2$. Then

\[ P(n^{\frac{1}{p} - \frac{1}{q}}(\sum_{i=1}^{n} x_i^q)^{1/q} / (\sum_{i=1}^{n} x_i^p)^{1/p} > t) =$
\[
P(\sum_{i=1}^{n} x_i^q > \frac{t^q }{n^{p-1}} ) \leq \sum_{i=1}^{n/2} P(x_i^* > t p_j^{1/q} S / n^{\frac{1}{p}-\frac{1}{q}} ) + P(\sum_{i=\frac{n}{2}+1}^{n} x_i^q > t^q S^q / 2n^{\frac{2}{p}-1} )
\]

where \( \{ x_j^* \} \) denotes the nonincreasing rearrangement of \( \{|x_j|\} \).

Since
\[
\sum_{j=\frac{n}{2}+1}^{n} x_i^* q \leq \frac{n}{2} x^*_q \leq \frac{n}{2}(\sum_{i=1}^{n/2} x_i^*)^{q/p}
\]

\[
\leq 2^{\frac{2}{p}-1} S^q / n^{\frac{2}{p}-1}
\]

we get that if \( t \geq 2^{1/p} \), the second term in (3) is zero.

To evaluate the first term in (3) fix \( 1 \leq j \leq n/2 \). Then
\[
P(x_j^* > t p_j^{1/q} S / n^{\frac{1}{p}-\frac{1}{q}} ) \leq \left( \begin{array}{c} n \\ j \end{array} \right) P(x_1, \ldots, x_j > t p_j^{1/q} S / n^{\frac{1}{p}-\frac{1}{q}} )
\]

\[
\leq \left( \begin{array}{c} n \\ j \end{array} \right) P(x_1^p, \ldots, x_j^p > t p_j p^{p/q} \sum_{i=j+1}^{n} x_i^p / n^{1-p/q} ).
\]

From Lemma 2 (first 3 and then 2) we get that the last expression is dominated by
\[
\left( \begin{array}{c} n \\ j \end{array} \right) C^j \text{E} \exp(-j p_j^{p/q} t p \sum_{i=j+1}^{n} x_i^p / n^{1-p/q})
\]

\[
\leq \left( \begin{array}{c} n \\ j \end{array} \right) C^j \exp(-j p_j^{p/q} t p (n-j) / 2pn^{1-p/q})
\]

for some universal \( C \). Note that the last inequality holds if \( j n^{p/q-1} p_j^{p/q} t p \leq 1 \). If this is not the case the probability we are trying to evaluate is zero. Finally the last term is dominated by
\[
\exp\left( j (\log \frac{e n}{j} + C - \frac{p_j^{p/q} t p n^{p/q}}{4p}) \right), 
\]

(4)

Now let \( \alpha \) to be chosen momentarily. Let \( p_j \Gamma j = 1, \ldots, n/2 \) be such that
\[
j \left( \log \frac{e n}{j} + C - \frac{p_j^{p/q} p_{p/q}}{4p} \right) = -\alpha n^{p/q} q^{p}
\]
i.e.
\[
p_j = \left( 4p \log \frac{e n}{j} + \frac{4C p}{p_{p/q}} + \frac{4p}{j} \right)^{q/p}.
\]
We thus get that for some universal constant \( C \Gamma \)
\[
p_j \leq 2^{\alpha - 1} \left( \frac{(C p)^{q/p} (\log \frac{e n}{j})^{q/p}}{q} \right) + 2^{\alpha - 1} \frac{\alpha^{q/p} (4p)^{q/p}}{j q/p}.
\]
It is easy to see that for \( 1 \leq p < q < \infty \Gamma \)
\[
\sum_{j=1}^{n/2} (\log \frac{e n}{j})^{q/p} \leq A n \min \{ q^{1/p}, (\log n)^{1/p} \}
\]
for some universal \( A \). Thus the sum over \( j \) of the first terms in (5) is smaller than \( 1/4 \) if for some universal \( \gamma \Gamma t > \gamma \min \{ q^{1/p}, (\log n)^{1/p} \} \). The sum over \( j \) of the second terms in (5) is bounded by \( 1/4 \) if \( \alpha < B^{\frac{1}{p}} (\frac{q}{p} - 1)^{p/q} \Gamma \) for some universal \( B \). Choosing \( \alpha \) to satisfy this inequality and using (3)\( \Pi \)(4) and (5) we get that for \( t > \gamma \min \{ q^{1/p}, (\log n)^{1/p} \} \Gamma \)
\[
\mu(\|u\|_q > t) \leq \frac{n}{2} e^{-\alpha n^{p/q} q^{p}}.
\]
Under the conditions on \( t \) the factor \( n/2 \) can be absorbed in the second term (changing \( \alpha \) to another constant of the same order of magnitude as a function of \( p \) thus proving (1).

We now turn to the proof of the lower bound (2) which is simpler. Using Claim 1 again
\[
\mu(\|u\|_q > t) = P \left( n^{\frac{1}{p} - \frac{1}{q}} \left( \sum_{i=1}^{n} x_i^q \right)^{1/q} / \left( \sum_{i=1}^{n} x_i^p \right)^{1/p} > t \right)
\]
\[
\geq P \left( x_1 > St / n^{\frac{1}{p} - \frac{1}{q}} \right)
\]
\[
= P \left( x_1 > \frac{t}{(n(1-p/q) - t)^{1/p}} \left( \sum_{i=2}^{n} x_i^p \right)^{1/p} \right).
\]
Since \( t^p \leq \frac{1}{2} n^{(1-p/q)} \Gamma \) this dominates

\[
P \left( x_1 > \frac{2^{1/p} t}{1 + \frac{1}{q}} \left( \sum_{i=2}^{n} x_i^p \right)^{1/p} \right).
\]

Now by Claim 2.3. \( \Gamma \)

\[
P \left( x_1 > \frac{2^{1/p} t}{1 + \frac{1}{q}} \left( \sum_{i=2}^{n} x_i^p \right)^{1/p} \right) \geq \frac{c_p}{2p} \exp(-4t^p \sum_{i=2}^{n} x_i^p / n^{(1-p/q)})
\]

\[
= \frac{c_p}{2p} (\exp(-4t^p x_1^p / n^{(1-p/q)}))^{n-1}
\]

\[
= \frac{c_p}{2p} \left( 1 + \frac{4t^p}{n^{1-p/q}} \right)^{(n-1)/p} \quad \text{(by Claim 2.2.)}
\]

\[
\geq \frac{c_p}{2p} \exp \left( -\frac{4t^p(n-1)}{pn^{(1-p/q)}} \right)
\]

\[
\geq \frac{c_p}{2p} e^{4t^p n^{(1-p)/q}}.
\]

Finally observe that since \( c_p \) is bounded away from zero and \( t \geq 2 \Gamma \) the factor \( \frac{c_p}{2p} \) can be absorbed in the second term (changing 4 to another universal constant).

\[
\]

**Remarks:**

1. It follows from the proof that \( \Gamma \) for \( n \) large enough and \( q \) close to \( p \) one can take

\[
c(p, q) = \frac{c}{p} \left( \frac{q}{p} - 1 \right)
\]

for some universal constant \( c \).

2. It follows from the statement of the theorem that \( \Gamma \) for \( q = \infty \Gamma \)

\[
\mu(\|u\|_\infty > t) \leq e^{-\gamma t^p / p}
\]

for all \( t > \tau (\log n)^{1/p} \Gamma \) and

\[
\mu(\|u\|_\infty > t) \geq e^{-\Gamma t^p / p}
\]

for all \( 2 \leq t \leq \frac{1}{2} n^{1/p} \Gamma \) where \( \gamma \Gamma \) and \( \tau \) are universal constants.

3. Note that it follows from Claim 1 and Claim 2.4. that the order of magnitude of \( T \) is the correct one.
4. The restriction $p \geq 1$ in Theorem 3 above and in Corollary 4 below can be replaced by $p > 0$ if one replaces the inequality $t \geq 2$ with $t \geq d\Gamma$ for some $d$ depending only on $p$ and $q\Gamma$ and removes the “moreover” part. We didn’t check the dependence of the constants on $p$ and $q$ in this case.

The last remark is that one can get a similar statement for the full balls. We state it as a corollary.

**Corollary 4.** For all $1 \leq p < q < \infty$ there are constants $c = c(p,q)$, $C = C(p,q)$ and $T = T(p,q)$ such that if $\nu$ denotes the normalized Lebesgue measure on the ball of $L^n_p$ then, for all $n$ large enough,

$$\nu(\|u\|_{L^n_q} > t) \leq \exp(-ct^{p} n^{p/q})$$

for all $t > T$, and

$$\nu(\|u\|_{L^n_q} > t) \geq \exp(-Ct^{p} n^{p/q})$$

for all $2 \leq t \leq \frac{1}{\gamma} n^{\frac{1}{p} - \frac{1}{q}}$. Moreover, for $q > 2p$ (or any other universal positive multiple of $p$), one can take $c(p,q) = \frac{2}{p}$, $C(p,q) = \frac{\Gamma}{p}$ and $T(p,q) = \tau \min\{q, \log n\}^{1/p} \leq q^{1/p}$, where $\gamma, \Gamma$ and $\tau$ are universal constants.

The proof follows easily from Theorem 3 and the formula

$$\nu(A) = n \int_0^1 r^{n-1} \mu\left(\frac{A}{r}\right) dr$$

which holds for all Borel sets $A$ in the ball of $L^n_p$. 

9
References


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