Quiz 10A (Answers)

1. Let \( A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \). This problem concerns the initial value problem

\[
(1) \quad x'(t) = Ax(t), \quad x(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.
\]

(a) Find the eigenvalues of \( A \).
(b) Find an eigenvector belonging to each eigenvalue of \( A \).
(c) Using your answers in (a) and (b), find a fundamental set of solutions to (1).
(d) Find the solution to (1) using your answer to (c).

Answer. The characteristic equation of \( A \) is \( r^2 - 4 = 0 \). The eigenvalues of \( A \) are \( r = 2, -2 \).
An eigenvector belonging to 2 is \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), and an eigenvector belonging to -2 is \( \begin{bmatrix} -1 \\ 1 \end{bmatrix} \).

Then \( \{x_1, x_2\} \) is a fundamental set of solutions to (1), where \( x_1 = e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) and \( x_2 = e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \).

To satisfy the initial condition, one solves

\[
c_1 x_1(0) + c_2 x_2(0) = \begin{bmatrix} c_1 - c_2 \\ c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}
\]

and conclude that \( x(t) = x_1(t) + x_2(t) = \begin{bmatrix} e^{2t} - e^{-2t} \\ e^{2t} + e^{-2t} \end{bmatrix} \).