Homework Assignment 2

1. We wish to offer two proofs of the statement that any metric is equivalent to some bounded metric. Let \((X, d)\) be a metric space.

   (a) Show that \(\rho(x, y) := \frac{d(x, y)}{1 + d(x, y)}\) is a bounded metric in \(X\) and that \(\rho\) and \(d\) are equivalent metrics. In other words, they induce the same topology on \(X\).

   (b) Show that \(\sigma(x, y) := \min\{1, d(x, y)\}\) defines a metric in \(X\) which is equivalent to \(d\).

2. Let \(X\) be any set, and define \(T = \{U \subseteq X : X - U \text{ is countable}\} \cup \{X, \emptyset\}\). Show that \(T\) is a topology on \(X\).

3. Recall that a map \(f : X \to Y\) between two topological spaces is open if whenever \(U \subseteq X\) is open, then \(f(U) \subseteq Y\) is open in \(Y\). Consider \(X = \mathbb{R}^2\) with the lexicographic (dictionary) order (cf. p. 6, example 6 in the text) and \(\mathbb{R}\) with its usual topology. Let \(f : X \to \mathbb{R}\) and \(g : X \to \mathbb{R}\) be the projection onto the first and second factors, respectively. (That is, \(f(x, y) = x\) and \(g(x, y) = y\).) Show that \(f\) is not an open map and \(g\) is an open map.

4. Let \(D\) denote the unit ball \(B_1(0)\) centered at \(0 \in \mathbb{R}^2\) with the Euclidean metric.

   (a) Show that \(D\) is homeomorphic to \(\mathbb{R}^2\).

   (b) Show that \(D\) is homeomorphic to the upper half plane \(H = \{(x, y) \in \mathbb{R}^2 : y > 0\}\).

5. Looped Line Space. Let \(B\) be the collection of subsets of \(\mathbb{R}\) which are

   (i) either intervals of the form \((a, b)\) not containing 0,

   (ii) or subsets of the form \((-\infty, -n) \cup (-\epsilon, \epsilon) \cup (n, \infty)\) for all possible choices of \(\epsilon > 0\) and \(n \in \mathbb{Z}\).

   This space is called the \textit{looped line}. Verify that \(B\) forms a basis for a topology on the real line. Specifically show that if \(U, V \in B\), then \(U \cap V\) is the union of sets in \(B\).

6. Sliced Pie Space. Let \(B\) be the collection of all sets \(U_{\epsilon; L_1, \ldots, L_r}(x) \subset \mathbb{R}^2\) corresponding to all \(x \in \mathbb{R}^2, \epsilon > 0\), and finite collections of lines \(L_1, \ldots, L_r\) passing through \(x\), where

   \[ U_{\epsilon; L_1, \ldots, L_r}(x) = (B_\epsilon(x) - \{L_1, \ldots, L_r\}) \cup \{x\} \]

   Show that \(B\) forms a basis for a topology on \(\mathbb{R}^2\).
7. Solve problem 2, p. 3 in the text.

8. Solve problem 3, p. 3 in the text.