

Homework Assignment 3

1. Let $X$ be a topological space. Recall that a set $A \subset X$ is said to be nowhere dense in $X$ if $\overline{A}$ contains no nonempty open sets.

   (a) Let $U \subset X$ be open. Show that the boundary of $U$ is closed and nowhere dense in $X$.
   
   (b) Conversely, show that every closed, nowhere dense set is the boundary of an open set.

2. An open subset $U$ in a topological space is said to be regularly open if $U$ is the interior of its closure. A closed set is regularly closed is it is the closure of its interior. Show:

   (a) The complement of a regularly open set is regularly closed, and vice versa.
   
   (b) There are open sets in $\mathbb{R}$ (with the usual Euclidean topology) which are not regularly open.
   
   (c) If $A$ is any subset of a topological space, then $\text{int}(\overline{A})$ (the interior of the closure of $A$) is regularly open.

3. Let $X$ be the Sliced Pie space (Assignment #2, problem 6).

   (a) Given any line $L$ in $\mathbb{R}^2$, describe the subspace topology that one gets on $L$ from this new topology.
   
   (b) Show that Sliced Pie topology is not first countable.

4. Let $\mathbb{R}_l$ denote the real numbers under the lower limit topology (the left-closed, right-open interval topology). Describe the closure in $\mathbb{R}_l$ of the following sets:

   (a) $X = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$.
   
   (b) $X = \left\{ -\frac{1}{n} : n \in \mathbb{N} \right\}$.
   
   (c) $X = \mathbb{Q}$, the set of rational numbers.

5. Given a topological space $X$ and a subset $A \subset X$, one says that $x_0 \in X$ is an accumulation point of $A$ if every neighborhood $U$ of $x_0$ contains a point of $A$ other than $x_0$.

   (a) Show that $A$ is closed in $X$ if and only if it contains all of its accumulation points.
   
   (b) Show that the closure of $A$ is the union of $A$ with the set of its accumulation points.
6. Let $\mathbb{R}_\mathbb{Z}^2$ denote the plane with the Zariski topology. A basis $\mathcal{B}$ for $\mathbb{R}_\mathbb{Z}^2$ consists of all subsets of $\mathbb{R}^2$ whose complements are zero sets of polynomials with real coefficients. That is, $U$ is a basic open set for $\mathbb{R}_\mathbb{Z}^2$ if there is a polynomial $p : \mathbb{R}^2 \to \mathbb{R}$ with real coefficients such that $U = \mathbb{R}^2 - \{(x, y) : p(x, y) = 0\} = \{(x, y) : |p(x, y)| > 0\}$.

(a) Show that $\mathcal{B}$ is a basis for a topology; i.e., show that $\mathcal{B}$ is closed under finite intersections.

(b) Let $f(x, y)$ and $g(x, y)$ be two polynomials with real coefficients, and define a function $F : \mathbb{R}_\mathbb{Z}^2 \to \mathbb{R}_\mathbb{Z}^2$ by $F(x, y) = (f(x, y), g(x, y))$. Show that $F$ is continuous in the Zariski topology.