Assignment 3
Solutions to Selected Problems

1. Let \( X \) be a topological space. Recall that a set \( A \subseteq X \) is said to be nowhere dense in \( X \) if \( \overline{A} \) contains no nonempty open sets.

   (a) Let \( U \subseteq X \) be open. Show that the boundary of \( U \) is closed and nowhere dense in \( X \).
   
   (b) Conversely, show that every closed, nowhere dense set is the boundary of an open set.

**Answer.** Regarding (a), \( \partial U = \overline{U} - U \) is closed. Furthermore, if \( W \subseteq \partial U \) is an open neighborhood of \( x \), then \( x \notin U \) by definition.

Regarding (b), let \( F \) be a closed, nowhere dense set, and \( U = X - F \). Clearly \( U \) is open and \( \overline{U} - U \subseteq F \). On the other hand, \( F - \overline{U} = \emptyset \) since \( F \) is nowhere dense.

2. An open subset \( U \) in a topological space is said to be regularly open if \( U \) is the interior of its closure. A closed set is regularly closed if it is the closure of its interior. Show:

   (a) The complement of a regularly open set is regularly closed, and vice versa.
   
   (b) There are open sets in \( \mathbb{R} \) (with the usual Euclidean topology) which are not regularly open.
   
   (c) If \( A \) is any subset of a topological space, then \( \text{int}(\overline{A}) \) (the interior of the closure of \( A \)) is regularly open.

**Answer.** For (a) and (c), review the solution of problem 1. For (b), consider \((-1,0) \cup (0,1)\).

3. Let \( X \) be the Sliced Pie space (Assignment #2, problem 6).

   (a) Given any line \( L \) in \( \mathbb{R}^2 \), describe the subspace topology that one gets on \( L \) from this new topology.
   
   (b) Show that Sliced Pie topology is not first countable.

**Answer.** (a) The subspace topology is the discrete topology.

(b) Let \( S \) be the intersection of a countable collection of sets in \( B \) centered at \( x \). Then \( S \) will omit at most a countable number of lines thought \( x \). Any set \( U \in B \) centered at \( x \) that omits some other line cannot be the union of sets in \( B \) centered at \( x \).

4. Let \( \mathbb{R}_* \) denote the real numbers under the lower limit topology (the left-closed, right-open interval topology). Describe the closure in \( \mathbb{R}_* \) of the following sets:
(a) \( X = \{ \frac{1}{n} : n \in \mathbb{N} \} \).

(b) \( X = \{-\frac{1}{n} : n \in \mathbb{N} \} \).

(c) \( X = \mathbb{Q} \), the set of rational numbers.

**Answer.** (a) \( X \cup \{0\} \) (b) \( X \) (c) \( \mathbb{R}_\ell \)

5. Given a topological space \( X \) and a subset \( A \subset X \), one says that \( x_0 \in X \) is an *accumulation point* of \( A \) if every neighborhood \( U \) of \( x_0 \) contains a point of \( A \) other than \( x_0 \).

(a) Show that \( A \) is closed in \( X \) if and only if it contains all of its accumulation points.

(b) Show that the closure of \( A \) is the union of \( A \) with the set of its accumulation points.

**Answer.** (a) If \( A \) is closed, then \( X - A \) is open and no point of \( X - A \) can be an accumulation point of \( A \). On the other hand, if \( A \) is not closed, then \( X - A \) is not open and there exists \( x \in X - A \) such that every neighborhood of \( x \) intersects \( A \). That is, \( x \) is an accumulation point of \( A \) not in \( A \).

(b) Let \( A' \) denote the accumulation points of \( A \). \( A \cup A' \) is closed by (a), so \( A \subset \overline{A} \subset A \cup A' \). However, arguing as in (a), \( A' - \overline{A} = A' \cap (X - \overline{A}) = \emptyset \).

6. Let \( \mathbb{R}_Z^2 \) denote the plane with the *Zariski topology*. A basis \( \mathcal{B} \) for \( \mathbb{R}_Z^2 \) consists of all subsets of \( \mathbb{R}^2 \) whose complements are zero sets of polynomials with real coefficients. That is, \( U \) is a basic open set for \( \mathbb{R}_Z^2 \) if there is a polynomial \( p : \mathbb{R}^2 \to \mathbb{R} \) with real coefficients such that \( U = \mathbb{R}^2 - \{(x, y) : p(x, y) = 0\} = \{(x, y) : |p(x, y)| > 0\} \).

(a) Show that \( \mathcal{B} \) is a basis for a topology; i.e., show that \( \mathcal{B} \) is closed under finite intersections.

(b) Let \( f(x, y) \) and \( g(x, y) \) be two polynomials with real coefficients, and define a function \( F : \mathbb{R}_Z^2 \to \mathbb{R}_Z^2 \) by \( F(x, y) = (f(x, y), g(x, y)) \). Show that \( F \) is continuous in the Zariski topology.

**Answer.** For (a), given two polynomials, the intersection of their zero sets is the zero set of their product.

For (b), if \( C \) is the zero set of \( h(x, y) \), then \( F^{-1}(C) \) is the zero set of \( h \circ F \).