On the entropy of actions of nilpotent Lie groups and their lattice subgroups

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Abstract

We consider a connected simply connected Lie group $G$ containing a lattice subgroup $\Gamma$. We prove the existence of a compact subset $C(\Gamma)$ of $G$ with properties: $C(\Gamma)$ tiles $G$ with tiling centers $\Gamma$, there is a lattice subgroup $\Gamma'$ such that $\Gamma \subset \Gamma'$, $[\Gamma' : \Gamma] = n$ for some natural $n$, and furthermore, $|C(\Gamma')| = n^{-1}|C(\Gamma)|$ where $|C(\Gamma)|$ is the Haar measure of $C(\Gamma)$. Let now $T$ be an ergodic action of $G$ on a probability space $(X, \mathcal{B}, \mu)$ preserving measure $\mu$, and $T^\Gamma$, the restriction of $T$ to $\Gamma$ is also ergodic. Then the following formula: $h(T) = |C(\Gamma)|^{-1}h_K(T^\Gamma)$ holds, where $h(T)$ is the Ornstein-Weiss entropy of $T$ and $h(T^\Gamma)$ is Kolmogorov-Sinai entropy of $T^\Gamma$.

If $T^\Gamma$ is not ergodic on $(X, \mathcal{B}, \mu)$ for some lattice subgroup $\Gamma$ of $G$ then there is a $T$-invariant subfactor $(Y, \mathcal{B}_Y, \nu)$ such that the restriction $T_Y$ of $T$ to $Y$ is reduced to transitive action of the quotient group $G/(K\Gamma)$ where $K$ is a commutant of $G$. We prove that $T$ has completely positive entropy (CPE) if and only if $T^\Gamma$ has CPE for some lattice subgroup $\Gamma$ of $G$. It is possible to deduce from this that $T$ has CPE if and only if $T$ is uniformly mixing. In this case $T$ has Lebesgue spectrum with infinite multiplicity.

Let $T$ be ergodic free action of $G$ on $X$ as above with a positive entropy. It is introduced the notion of the Pinsker algebra $\Pi(T)$ for the action $T$, and let $\Gamma$ be any lattice subgroup of $G$ that $T^\Gamma$ is ergodic and the Pinsker algebra $\Pi(T^\Gamma)$ is not $\nu$. Then we show $\Pi(T^\Gamma)$ is a $T$-invariant $\sigma$-subalgebra of $\mathcal{B}$ and $\Pi(T) = \Pi(T^\Gamma)$. If $\Gamma'$ is any other ergodic lattice subgroup of $G$ then again $\Pi(T^\Gamma) = \Pi(T)$. In this case, $T$ always has Lebesgue spectrum with infinite multiplicity on the space $L^2(X, \mu) \ominus L^2(\Pi(T))$ where $L^2(\Pi(T))$ contains all $\Pi(T)$-measurable functions from $L^2(X, \mu)$.

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