LIOUVILLE THEOREMS AND SPECTRAL EDGE
BEHAVIOR ON ABELIAN COVERINGS OF COMPACT
MANIFOLDS

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Abstract. The paper describes relations between Liouville type
theorems for solutions of a periodic elliptic equation (or a system)
on an abelian cover of a compact Riemannian manifold and the
structure of the dispersion relation for this equation at the edges
of the spectrum. Here one says that the Liouville theorem holds
if the space of solutions of any given polynomial growth is finite
dimensional. The necessary and sufficient condition for a Liouville
type theorem to hold is that the real Fermi surface of the elliptic
operator consists of finitely many points (modulo the reciprocal
lattice). Thus, such a theorem generically is expected to hold at
the edges of the spectrum. The precise description of the spaces
of polynomially growing solutions depends upon a ‘homogenized’
constant coefficient operator determined by the analytic structure
of the dispersion relation. In most cases, simple explicit formulas
are found for the dimensions of the spaces of polynomially growing
solutions in terms of the dispersion curves. The role of the base of
the covering (in particular its dimension) is rather limited, while
the deck group is of the most importance.

The results are also established for overdetermined elliptic sys-
tems, which in particular leads to Liouville theorems for polynomi-
ally growing holomorphic functions on abelian coverings of compact
analytic manifolds.

Analogous theorems hold for abelian coverings of compact combi-
natorial or quantum graphs.

1. Introduction

The classical Liouville theorem claims that any harmonic function
(i.e., a solution of the Laplace equation \( \Delta u = 0 \)) in \( \mathbb{R}^n \) that has a
polynomial upper bound is in fact a (harmonic) polynomial. In par-
icular, the space of all harmonic functions that grow not faster than

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theorem, periodic operator, abelian cover, holomorphic function.} \]
$C(1 + |x|)^N$, is of the finite dimension\footnote{We will also use the notation}

\begin{equation}
(1.2) \quad h_{n,N} := \left( \frac{n + N}{N} \right) - \left( \frac{n + N - 2}{N - 2} \right).
\end{equation}

The problem of extending this result to more general elliptic operators and/or to Laplace-Beltrami operators on general Riemannian manifolds of non-negative Ricci curvature\footnote{Without a condition on the curvature, the hyperbolic plane, where there is an infinite dimensional space of bounded harmonic functions, provides a counterexample.} has gained prominence since the work of S. T. Yau [85]. The questions asked concern finite dimensionality of the spaces of solutions of a prescribed polynomial growth, estimates of (and in rare cases formulas for) their dimensions, and structure of these solutions. One can find recent advances, reviews, and references in [24, 58, 59]. In particular, the Yau's conjecture on the validity of the Liouville theorem for Riemannian manifolds of non-negative Ricci curvature was proven in full generality in [24] (see also references to previous partial solutions in [58, 59]).

In the flat situation, an amazing case was discovered by M. Avelaneda and F.-H. Lin [5] and later also studied by J. Moser and M. Struwe [67]. In these papers the authors dealt with polynomially growing solutions of second-order divergence form elliptic equations

\begin{equation}
(1.3) \quad Lu = - \sum_{1 \leq i,j \leq n} (a^{ij}(x)u_{x_i})_{x_j} = 0
\end{equation}

with coefficients that are periodic with respect to the lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$. For those equations they obtained a comprehensive answer (see also [24, 58] for related results and references). Using the formalism of homogenization theory, it was proved that the space of all solutions of the equation $Lu = 0$ of polynomial growth of order at most $N$ has the same dimension $h_{n,N}$ (see (1.2)) as the space of harmonic polynomials in $\mathbb{R}^n$ of the same rate of growth. Moreover, any solution $v$ of the equation $Lv = 0$ in $\mathbb{R}^n$ of polynomial growth is representable as a finite sum of the form

\begin{equation}
(1.4) \quad v(x) = \sum_{j=(j_1, \ldots, j_n) \in \mathbb{Z}^n_+} x^j p_j(x),
\end{equation}

for the dimension of the space of all polynomials of degree at most $N$ in $n$ variables. Notice that $q_{n,N}$ also coincides with the dimension of the space of all homogeneous polynomials of degree $N$ in $n$ variables, so in particular, $h_{n,N} = q_{n,N} - q_{n,N-2} = q_{n-1,N-1} + q_{n-1,N}$.\footnote{We will also use the notation.
where the functions $p_j(x)$ are periodic with respect to the group of periods of the equation.

One can say that there has been no complete understanding of this result on periodic equations. In particular, one can ask the following natural questions: (i) Is it important that the operator is of divergence form? (ii) Can the results be generalized to higher order equations? (iii) Is it possible to determine for a given periodic elliptic equation whether the Liouville theorem holds? (iv) How crucial is the usage of homogenization theory tools (which automatically restricts the class of equations)? (v) Same questions about elliptic systems. (vi) Can these results be generalized for covering spaces of compact manifolds?

Some partial answers to these questions were obtained in [57, 60]. In [60], the results for the divergence type operators (1.3) in $\mathbb{R}^n$ were generalized to the case of second-order periodic operators without lower order terms. At the same time, [57] contained a necessary and sufficient condition for the validity of the Liouville theorem for a general periodic elliptic operator in $\mathbb{R}^n$, as well as (in most cases implicit) description of the dimensions of the corresponding spaces of solutions. In particular, an explicit formula was given for a general second-order periodic elliptic operator that admits a global positive solution.

Simultaneously with this kind of studies, an activity has existed of studying Liouville theorems for holomorphic functions on complex analytic manifolds (see [42, 61, 63, 64] and references therein). In particular, one asks whether Liouville theorems for holomorphic functions hold for coverings of compact analytic manifolds (or more generally, of manifolds with the Liouville property). One should mention the results of [61], where it was shown in particular that nilpotent coverings of compact complex analytic manifolds do have the Liouville property for bounded holomorphic functions (i.e., the space of such functions is finite-dimensional). It was not clear whether one could say the same about the spaces of functions of a given polynomial growth. The exception was the result of [19], where this was proven for abelian coverings of compact Kähler manifolds (see also [20]). This result also follows from [24], while this relation with Liouville theorems for harmonic functions disappears for non-Kähler case.

One should also mention another parallel activity that concerned with Liouville theorems for harmonic functions on graphs (e.g., [42, 66]).

The goal of this paper is to provide results of Liouville type that clarify this issue for abelian covers of compact manifolds. The results apply to elliptic equations and systems (including overdetermined ones) on abelian coverings of compact Riemannian manifolds, as well as to
holomorphic functions on abelian coverings of compact complex manifolds, and to periodic equations on abelian coverings of combinatorial and quantum graphs. The crucial techniques used in the paper are different from the ones used in all the works cited above, with the exception of the authors’ paper [57]. They come from the Floquet theory [50, 75] and are related to some spectral notions common in the solid state physics [7] (the reader can find all necessary preliminary information in the next two sections). In comparison with [57], the results of this paper provide explicit dimensions where [57] only contained implicit constructions, multiplicities at spectral edges are allowed, overdetermined systems (in particular, \( \partial \)-operators and Liouville theorems for holomorphic functions), as well as graph operators are considered. General theorems are applied to a large set of specific examples of operators.

In order to outline the results of the paper, let us introduce some objects first. Consider a normal abelian covering of a compact \( d \)-dimensional Riemannian manifold \( M \)

\[
X \overset{G}{\rightarrow} M,
\]

where \( G \) is the (abelian) deck group of the covering. Without loss of generality one can assume that \( G = \mathbb{Z}^n \). In fact, no harm will be done if the reader imagines for simplicity that \( X = \mathbb{R}^n \), \( G = \mathbb{Z}^n \), and \( M \) is the torus \( \mathbb{R}^n / \mathbb{Z}^n \) (albeit in general the dimension of \( M \) does not have to be equal to \( n \)).

We will need to consider characters \( \chi \) of \( G \), i.e. homomorphisms of \( G \) into the multiplicative group \( \mathbb{C}^* \) of non-zero complex numbers. Unitary characters map \( G \) into the group \( S^1 \) of complex numbers of absolute value 1. For any character \( \chi \), a function \( f \) on \( X \) will be called \( \chi \)-automorphic, if \( f(gx) = \chi(g)f(x) \) for any \( x \in X \), \( g \in G \).

Let now \( P \) be an elliptic \( G \)-periodic operator on \( X \) (in what follows, we will use for shortness the word ‘periodic’ instead of ‘\( G \)-periodic’). For any character \( \chi \), consider the space of \( \chi \)-automorphic functions on \( X \). It can also be interpreted as the space of sections of a linear bundle over \( M \) determined by \( \chi \). It is invariant with respect to \( P \), so one can consider the restriction \( P(\chi) \) of \( P \) to this space\(^4\). The spectrum of \( P(\chi) \), as a multiple-valued function of the character \( \chi \), is said to be the dispersion curve or dispersion relation. In the particular case of \( \chi(g) \equiv 1 \), \( P(1) \) is just the elliptic operator \( P_M \) on \( M \), whose lifting to

\(^3\)The word ‘covering’ in this text always means ‘normal covering’.

\(^4\)Exact definitions of function spaces and operators are provided in the next section.
$X$ is $P$. In ‘non-pathological’ cases, the spectra of all operators $P(\chi)$ are discrete.

We will say that the Liouville theorem holds to an order $N$ for the equation $Pu = 0$, if the space $V_N(P)$ of solutions of the equation that have a bound $|u(x)| \leq C(1 + \rho(x))^N$ is finite dimensional. Here $\rho(x)$ is the distance of $x \in X$ from a fixed point $x_0 \in X$.

We can now formulate a general (and somewhat vague at this point) statement that outlines our main results for the elliptic case contained in Theorems 17, 18, and 28. The results for the overdetermined, holomorphic, and graph cases can be found in Sections 6 and 7.

**Main Theorem.**

1. If the Liouville theorem for the equation $Pu = 0$ holds to an order $N \geq 0$, it holds to any order.
2. In order for the Liouville theorem to hold, it is necessary and sufficient that the number of unitary characters $\chi$ for which the equation $Pu = 0$ has a non-zero $\chi$-automorphic solution is finite.
3. If the Liouville theorem holds and the spectra of all operators $P(\chi)$ are discrete, then under some genericity condition on the operator $P$, the dimension of the space $V_N(P)$ can be computed in terms of the dispersion relation for $P$.
4. Under the same conditions, one can describe a constant coefficient (‘homogenized’) linear differential operator $\Lambda(D)$ on $\mathbb{R}^n$, such that there is a one-to-one correspondence between polynomial solutions of $\Lambda v = 0$ on $\mathbb{R}^n$ and polynomially growing solutions of $Pu = 0$ on $X$.

We will see that this result in particular means that one should naturally expect the Liouville theorem to hold only when zero is at an edge of the spectrum of $P$. This is true, for instance, for the operators considered in [5, 60, 67], when zero is the bottom of the spectrum.

It is interesting to notice that the dimension of $X$ does not have to be equal to $n$, so the operators $P$ and $\Lambda$ might act on manifolds of different dimensions. This happens since the Liouville property is of a ‘homogenized’ nature, i.e. it is something one sees by looking at the manifold $X$ ‘from afar’. Thus, the local details of the manifold are essentially lost and one sees the euclidian space $\mathbb{R}^n$ instead. In other words, the free rank of the deck group of the covering plays a more prominent role for Liouville theorems than the dimension of the manifold\(^5\).

\(^5\)This resonates with M. Gromov’s notion of quasi-isometry, when the space of the covering might be indistinguishable from the deck group [33, 37].
Similar results hold for elliptic systems, including overdetermined ones that are elliptic in the sense of being a part of an elliptic complex of operators. The reader can find basic notions and results concerning elliptic complexes in many books and articles (e.g., [40, Vol. III, Section 19.4], [76, Section 3.2.3], [82, Section IV.5]). For the particular case of the Cauchy-Riemann operator, one obtains a Liouville theorem for holomorphic functions on abelian coverings (Theorem 29). Analogs for operators on combinatorial and quantum graphs are also straightforward to obtain.

The outline of the paper is as follows. The next section introduces necessary notations and preliminary results from the Floquet theory, in particular the definition and properties of the Floquet-Gelfand transform. The proofs are the same as for the case of periodic operators on $\mathbb{R}^d$ and are hence mostly omitted (see, e.g., how they can be worked out in parallel to the flat case in [47] and also in [21, 22, 81]). The crucial Section 3 is devoted to the detailed study of the so-called Floquet-Bloch solutions. In Section 4 we derive Liouville type theorems for elliptic systems. The next Section 5 provides some examples of applications to specific periodic operators. Section 6 treats overdetermined systems, including the case of analytic functions on complex manifolds. Graphs are briefly considered in Section 7. The last sections contain conclusions, remarks, and acknowledgments.

2. Notations and preliminary results

Let us introduce first some standard notions of Floquet theory (see [29, 50, 75]), which we will adjust to the case of abelian covers (this does not require any change in the substance).

Let $X$ be a non-compact smooth Riemannian manifold of dimension $d$ equipped with an isometric, properly discontinuous, and free action of a finitely generated abelian group $G$. The action of an element $g \in G$ on $x \in X$ will be denoted by $gx$. Consider the orbit space $M = X/G$, which due to our conditions is a Riemannian manifold of its own. We will assume that $M$ is compact. Hence, we are dealing with an abelian covering of a compact manifold

$$\pi : X \to M (= X/G).$$

Switching to a subcovering $X \to \tilde{M} \to M$ with a compact $\tilde{M}$, one can eliminate the torsion part of $G$. In what follows, we could substitute $\tilde{M}$ for $M$ and hence reduce the group $G$ to $\mathbb{Z}^n$ with some $n \in \mathbb{N}$. We will therefore assume from now on that $G = \mathbb{Z}^n$. This will not reduce the generality of the results.
Let $P$ be an elliptic operator of order $m$ on $X$ with smooth coefficients$^6$ that commutes with the action of $G$. Such an operator can be pushed down to an elliptic operator $P_M$ on $M$ (or conversely, $P$ is the lifting of $P_M$ to $X$). The ellipticity is understood in the sense of non-vanishing of the principal symbol of the operator $P$ on the co-tangent bundle (with the zero section removed) $T^*X \setminus (X \times \{0\})$. The dual operator (the formal adjoint) $P^*$ has similar properties (in particular, $P^*$ is also $G$-periodic). Here the duality is provided by the bilinear rather than the sesquilinear$^7$ form

$$< g, f > = \int_X f(x)g(x)dx$$

All the preparatory facts and main statements here hold for linear periodic matrix operators that are either standard elliptic (sometimes called elliptic in Petrovsky sense) or elliptic in the Douglis-Nirenberg sense (e.g., [28], [40, Vol. III, Section 19.5], and [76, Section 3.1.2.1]). The only difference in the proofs between the scalar and system cases arises in necessity of introducing appropriate spaces of vector-valued functions, exactly as it was done in [50, Section 3.4]. Doing this, however, would on one hand be very routine, and on the other hand would make reading the text more difficult. Bearing this in mind, we will provide detailed considerations for scalar linear elliptic partial differential operators only. They transfer with no effort to the systems.

For any quasimomentum $k \in \mathbb{C}^n$ we denote by $\gamma_k$ the character of $G = \mathbb{Z}^n$ defined as $\gamma_k(g) = e^{ik \cdot g}$ Here $g = (g_1, \ldots, g_n) \in \mathbb{Z}^n$ and $k \cdot g = k_1g_1 + \ldots + k_ng_n$. We will also use the notation $|g| = |g_1| + \ldots + |g_n|$ and for a multi-index $j = (j_1, \ldots, j_n) \in \mathbb{Z}^n_+$ we denote $g^j = \prod_i g_i^{j_i}$. If $k \in \mathbb{R}^n$, the corresponding character is unitary. Due to the obvious $2\pi$-periodicity of $\gamma_k$ with respect to $k$, it is sufficient to restrict ourselves to the real vectors $k$ in the Brillouin zone $B = [-\pi, \pi]^n$, which is a fundamental domain of the reciprocal (dual) lattice $G^* = (2\pi \mathbb{Z})^n$. Periodizing $B$ (i.e., considering the torus $\mathbb{T}^n = \mathbb{R}^n/2\pi \mathbb{Z}^n$), one obtains the dual group $\mathbb{T}^n$ to $G = \mathbb{Z}^n$.

For any $k \in \mathbb{C}^n$, we define the subspace $L_k^2(X)$ of $L^2_{loc}(X)$ consisting of all functions $f(x)$ that are $\gamma_k$-automorphic, i.e. such that $f(gx) = \gamma_k(g)f(x) = e^{ik \cdot g}f(x)$ for a.e. $x \in X$. Alternatively, this space

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$^6$The smoothness condition can be significantly reduced (see the corresponding remarks in Section 8, as well as Remark 6.1 in [57]). We only need that both the operator $P_M$ and its dual $P_M^*$ define Fredholm mappings between the Sobolev space $H^m(M)$ and $L^2(M)$, and this condition can be weakened further. Essentially, one needs to guarantee compactness of resolvent of $P_M$.

$^7$This is not essential, but simplifies somewhat the calculations.
can be defined as follows. We can identify $\gamma_k$ with a one-dimensional representation of $G$ and consider the one-dimensional flat vector bundle $E_k$ over $M$ associated with this representation. Then elements of $L^2_k(X)$ can be naturally identified with $L^2$-sections of $E_k$. A similar construction works also for other classes of functions (e.g., from Sobolev spaces). We will identify $G$-periodic (i.e., $\gamma_0$-automorphic) functions on $X$ with functions on $M$. Due to the periodicity of the operator $P$, it leaves the spaces $L^2_k$ invariant, and so its restrictions to these subspaces define elliptic operators $P(k)$ on the spaces of sections of the bundles $E_k$ over $M$.

It is natural that Fourier transform with respect to the periodicity group $G$ reduces the space $L^2(X)$, as well as the original operator $P$ on $X$, to the direct integral of operators $P(k)$ on sections of $E_k$:

\begin{equation}
L^2(X) = \bigoplus_B L^2_k(X) \, dk
\end{equation}

and

\begin{equation}
P = \bigoplus_B P(k) \, dk
\end{equation}

The integral is understood with respect to the normalized Haar measure on the dual group $T^n$, which boils down to the normalized Lebesgue measure $dk$ on the Brillouin zone $B$. The isomorphism (in fact, an isometry) in (2.2) is provided by an analog of the Fourier transform (see [50, Section 2.2], [75, 81]), which we will call the Floquet-Gelfand transform $\mathcal{U}$:

\begin{equation}
f(x) \rightarrow \mathcal{U}f(k, x) = \sum_{g \in G} f(gx) \gamma_{-k}(g), \quad k \in \mathbb{C}^n.
\end{equation}

This transform is the main tool in the Floquet theory for PDEs (e.g., [50, 75, 79, 81]). It was introduced first in [32] in order to obtain expansions into Bloch generalized eigenfunctions for periodic self-adjoint elliptic operators.

It is not hard to describe the image of a Sobolev space $H^s(X)$ under the Floquet-Gelfand transform. In order to do so, let us consider a quasimomentum $k \in \mathbb{C}^n$ and denote by $H^s_k$ the closed subspace of the

\footnote{In the case when $X = \mathbb{R}^n$ with the natural $\mathbb{Z}^n$ action, these operators can be identified with the “shifted” versions $P(x, D + k)$ of the operator $P$ acting on the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ (see [50, 75]).}
space $H_{loc}(X)$ consisting of $\gamma_k$-automorphic functions. It is clear that $H^s_k$ can be naturally equipped with the structure of a Hilbert space and that it can be identified with the space $H^s(E_k)$ of $H^s$-sections of the bundle $E_k$ over $M$.

One can show\(^9\) that

\[(2.5) \quad \mathcal{E}^s := \bigcup_{k \in \mathbb{C}^n} H^s_k\]

forms a holomorphic $2\pi \mathbb{Z}^n$-periodic bundle. As any infinite dimensional analytic Hilbert bundle over a Stein domain, it is trivializable [23] (see also the survey [86] and Theorems 1.3.2, 1.3.3, and 1.5.23 in [50]).

We collect now several statements from [75, Theorem XIII.97], [50, Theorem 2.2.2], and [57, 81], recasted into the abelian covering form:

**Theorem 1.**

1. For any nonnegative integer $m$, the operator
   \[ U : H^m(X) \to L^2(\mathbb{T}^n, \mathcal{E}^m) \]
   is an isometric isomorphism, where $L^2(\mathbb{T}^n, \mathcal{E}^m)$ denotes the space of square integrable sections over the torus (identified with the Brillouin zone $B$) of the bundle $\mathcal{E}^m$, equipped with the natural topology of a Hilbert space.

2. Let $K \subset X$ be a domain in $X$ such that $\bigcup gK = X$. Let also the space
   \[ C^m = \left\{ \phi \in H^m_{loc}(X) \left| \sup_{g \in G} \|\phi\|_{H^m(gK)} (1 + |g|)^N < \infty, \forall N \right. \right\} \]
   be equipped with the natural Fréchet topology. Then
   \[ U : C^m \to C^\infty(\mathbb{T}^n, \mathcal{E}^m) \]
   is a topological isomorphism, where $C^\infty(\mathbb{T}^n, \mathcal{E}^m)$ is the space of $C^\infty$ sections of the bundle $\mathcal{E}^m$ over the complex torus $\mathbb{T}^n$, equipped with the standard topology.

3. Let the elliptic operator $P$ be of order $m$. Then under the transform $U$ the operator
   \[ P : C^m(X) \to C^0(X) \]
   becomes the operator
   \[ C^\infty(\mathbb{T}^n, \mathcal{E}^m) \xrightarrow{P(k)} C^\infty(\mathbb{T}^n, \mathcal{E}^0) \]
   of multiplication by the holomorphic Fredholm morphism $P(k)$ between the fiber bundles $\mathcal{E}^m$ and $\mathcal{E}^0$.

\(^9\)See Theorem 2.2.1 in [50] for the case $X = \mathbb{R}^n$. The general case of abelian covers over compact manifolds is entirely parallel.
3. Floquet-Bloch solutions

We now need to introduce and to study our main notions: Bloch and Floquet solutions of periodic differential equations.

**Definition 2.** Let $k \in \mathbb{C}^n$. A $\gamma_k$-automorphic function $u(x)$ on $X$ is said to be a **Bloch function with quasimomentum** $k$. In other words, it is a function with the property $u(gx) = \gamma_k(g)u(x) = e^{ik \cdot g}u(x)$ for any $x \in X$, $g \in G$. Yet another way to put it is that $u(x)$ is transformed according to an irreducible representation of group $G$ with the character $\gamma_k$.

A **Bloch solution** of an equation is a solution that is a Bloch function.

Notice that every continuous Bloch function on $X$ with a real quasimomentum (i.e., transformed according to an irreducible unitary representation) is bounded. Any such Bloch function $u$ that belongs to $L^2_{\text{loc}}(X)$ is bounded in the following integral sense: for any compact $K \subset X$ we have $\sup_{g \in G} \|u\|_{L^2(gK)} < \infty$.

In the case when $X = \mathbb{R}^n$, $G = \mathbb{Z}^n$, and $M = \mathbb{T}^n$, any Bloch function with quasimomentum $k$ has the form $u(x) = e^{ik \cdot x}p(x)$ with a $\mathbb{Z}^n$-periodic function $p(x)$. In fact, a similar (albeit less natural) representation holds for Bloch functions on any abelian cover $X \twoheadrightarrow M$. Indeed, let $K$ be any fundamental domain of $X$ with respect to the action of $G$ and $f \in C_0^\infty(X)$ be a nonnegative function strictly positive on $K$. We define for any $j = 1, \ldots, n$

$$h_j(x) := \sum_{g=(g_1, \ldots, g_n) \in G=\mathbb{Z}^n} f(gx) \exp(-g_j).$$

Then $h_j(x)$ clearly is a positive function satisfying $h_j(gx) = e^{g_j}h_j(x)$ for any $g = (g_1, \ldots, g_n) \in G$. It is an analog of $e^{x_j}$ on $\mathbb{R}^n$. Thus, one can define analogs of powers $x^l = x_1^{l_1} \cdots x_n^{l_n}$ and of exponents $e^{ik \cdot x}$ as follows: for $l = (l_1, \ldots, l_n) \in \mathbb{Z}_+^n$ let

$$[x]^l := \prod_{j=1}^n (\log h_j(x))^{l_j}$$

and for any quasimomentum $k \in \mathbb{C}^n$

$$e_k(x) := \exp(i k_1 \log h_1(x) + \cdots + i k_n \log h_n(x)).$$

Notice that $e_k(x)$ is a non-vanishing Bloch function on $X$ with the quasimomentum $k$, positive for real $k$. Thus, any Bloch function $u$ on
X with this quasimomentum is given by
\[ u(x) = e_k(x)p(x), \]
where \( p(x) \) is \( G \)-periodic.

The construction of functions \( \log h_j \) can be described in a more invariant way. Consider a basis \( \omega_j \) of the space of closed differential 1-forms on \( M \) (modulo the exact ones) such that their lifts \( w_j \) to \( X \) are exact. According to De Rham’s theorem, this basis is finite. One can now achieve the same goals as before defining \( h_j(x) = \exp(\int_o^x w_j) \) for a fixed point \( o \in X \).

We can now define a more general class than Bloch functions.

**Definition 3.** A function \( u(x) \) on \( X \) is said to be a Floquet function with the quasimomentum \( k \in \mathbb{C}^n \), if it can be represented in the form

\[
(3.1) \quad u(x) = e_k(x) \left( \sum_{j = (j_1, \ldots, j_n) \in \mathbb{Z}_+^n, |j| \leq N} [x]^j p_j(x) \right),
\]

where functions \( p_j \) are \( G \)-periodic.

The number \( N \) in this representation will be called the order of the Floquet function.

A Floquet solution of an equation is a solution that is a Floquet function.

This definition is modelled closely after the notion of Floquet solution that is common in \( \mathbb{R}^n \) (e.g., [50]), where the formula is the same, one just replaces the ‘powers’ \([x]^j\) by the true powers \(x^j\). However, unlike the notion of a Bloch function, it lacks invariance. To alleviate this, we briefly address now a different way to define Bloch and Floquet functions (solutions) on abelian coverings.

For any \( g \in G \), let us define the first difference operator \( \Delta_g \) acting on functions on \( X \) as follows:

\[
(3.2) \quad \Delta_g u(x) = u(gx) - u(x).
\]

It is quite clear that a function \( u(x) \) is periodic (i.e., is a Bloch function with zero quasimomentum) if and only if it is annihilated by \( \Delta_g \) for any \( g \in G \). In fact, it is sufficient to check this property for any set \( \{g_j\} \) of generators of \( G \). One wonders whether one can check in a similar way whether a function is a Bloch function with a non-zero quasimomentum
and whether Floquet functions allow for similar tests. In order to get
the answer, we need to introduce a twisted version of the first difference, that depends of the quasimomentum:
\[
\Delta_{g;k}u(x) = \chi_{-k}(g)u(gx) - u(x) = e^{-ik\cdot gx}u(gx) - u(x).
\]

We also need to introduce iterated finite differences of order \(N\) with quasimomentum \(k\) as follows:
\[
\Delta_{g_1,\ldots,g_N;k} = \Delta_{g_1;k} \cdots \Delta_{g_N;k},
\]
where \(g_j \in G\) (it will always be sufficient to use only elements (maybe repeated) of a fixed set of generators of \(G\)).

We can now answer the question by proving the following

**Lemma 4.** A function \(u(x)\) on \(X\) is a Floquet function of order \(N\) with quasimomentum \(k\), if and only if it is annihilated by any difference of order \(N+1\) with quasimomentum \(k\): \(\Delta_{g_1,\ldots,g_{N+1};k}u = 0\) for any \(g_j \in G\) (albeit, choosing elements of a fixed set of generators is sufficient). In particular, it is a Floquet function of order \(N\) with quasimomentum 0, if and only if
\[
\Delta_{g_1} \cdots \Delta_{g_{N+1}}u = 0 \quad \text{for any } g_1, \ldots, g_{N+1} \in G.
\]

**Proof:** Let us provide the proof for the case \(k = 0\) first. As it has already been mentioned, the necessity and sufficiency of the condition (3.5) checks out easily for \(N = 0\), where it boils down to \(\Delta_g u = 0\) for all \(g \in G\), i.e. to periodicity of \(u\). Necessity for any \(N\) now follows by simple induction, if one takes into account the representation (3.1). Indeed, one computes that (using the same standard basis \(\{g_j\}\) of \(\mathbb{Z}^n\) as before)
\[
\Delta_{g_j}[x]^{(l_1,\ldots,l_n)} = I_j[x]^{(l_1,\ldots,l_j-1,\ldots,l_n)} + \text{lower order terms}.
\]
Here ‘lower order terms’ contain linear combinations of \([x]^l\)'s of strictly lower total degrees. Since the difference operators do not alter periodic functions, we obtain that any \(\Delta_{g_j}\) reduces Floquet functions of order \(N+1\) to the ones of order \(N\), which concludes the induction step of the proof of necessity for \(k = 0\).

Let us prove sufficiency. It also follows by induction with respect to the order of the Floquet function. We have already checked it for \(N = 0\). Assume that this has been proven for orders up to \(N\). Suppose that \(u\) satisfies \(\Delta_{g_1} \cdots \Delta_{g_{N+2}}u = 0\) for any \(g_1, \ldots, g_{N+2} \in G\). Take a set of generators \(g_j\) in \(G\). Then functions \(f_j := \Delta_{g_j}u\) satisfy the condition of the Lemma for the order \(N\). According to the induction hypothesis,
we conclude that

\[ f_j = \left( \sum_{l = (l_1, \ldots, l_n) \in \mathbb{Z}_+^n} [x]^l p_{l,j}(x) \right) \]

with periodic functions \( p_{l,j} \). We claim now that for any Floquet function \( f \) of order \( N \) (with \( k = 0 \)) there exists a Floquet function \( v \) of order \( N + 1 \) such that \( \Delta_{g_j} v = f \). Indeed, let us agree first of all that \( j = 1 \), which does not limit the generality of our consideration. It is sufficient to check the statement for \( f = [x]^l \), since the difference operator \( \Delta_1 \) does not change periodic coefficients. Now induction with respect to \( l_1 \) finishes the job. Namely, (3.6) provides a system of a triangular structure for recursively determining the coefficients of \( v \) such that \( \Delta_{g_1} v = f \). Now a Floquet function \( v_1 \) of order \( N + 1 \) exists such that \( \Delta_1 v_1 = \Delta_1 u \). This means that the function \( u_2 = u - v_1 \) is periodic with respect to the one-parametric subgroup generated by \( g_1 \). Since the condition of the Lemma is still satisfied for \( v_2 \) and since \( u_2 \) is \( g_1 \)-periodic, we can continue this process and find a Floquet function of order \( N + 1 \) that is \( g_1 \)-periodic and such that \( u_3 = u - v_1 - v_2 \) is periodic with respect to both \( g_1 \) and \( g_2 \). Continuing this process, we get the conclusion of the Lemma for \( k = 0 \).

Let us now prove the statement for an arbitrary value of the quasimomentum \( k \). According to the definition (3.1) of Floquet functions, any Floquet function \( u \) of order \( N \) with quasimomentum \( k \) has the form \( u = e_k(x)v \), where \( v \) is a Floquet function of order \( N \) with quasimomentum \( k = 0 \). A straightforward calculation shows that if any two functions are related as \( u = e_k(x)v \), one has \( \Delta_{g_k} u = e_k(x)\Delta_g v \). Thus, the statement of the lemma for an arbitrary quasimomentum \( k \) follows from the one for \( k = 0 \). \( \square \)

We have provided several different ways to interpret the notion of Floquet solutions: using explicit formulas analogous to the ones in \( \mathbb{R}^n \) (constructing analogs of coordinate functions and exponents by either explicit constructions, or by using some special differential forms on \( M \)), as well as in terms of some difference operators\(^{10}\). Another way to think of Floquet functions is to imagine the subspace \( E \) generated by the \( G \)-shifts of such a function as of a Jordan block for the action of \( G \). Thus, relations to indecomposable representations of \( G \) arise (\cite{72},

\(^{10}\)Difference operators approach has been successfully used in related studies of periodic equations and Liouville type problems for holomorphic functions in \cite{61, 62}. )
see also [9, Ch. 6, Sect. 3] concerning the structure of indecomposable representations of abelian groups).

**Remark 5.** Notice that any continuous Floquet function \( u(x) \) of order \( N \) with a real quasimomentum can be estimated on \( gK \) as \( |u(x)| \leq C_K(1 + |g|)^N \) for all \( g \in G \). In general, one needs to replace this grows estimate by an integral one, as it was done before for Bloch functions.

We now introduce a notion that plays a very important role in solid state physics, photonic crystal theory, as well as in the general theory of periodic PDEs [7, 50, 51]. It will be also crucial for formulation of our main results.

**Definition 6.** The (complex) **Fermi surface** \( F_P \) of the operator \( P \) (at the zero energy level) consists of all quasimomenta \( k \in \mathbb{C}^n \) such that the equation \( Pu = 0 \) on \( X \) has a nonzero Bloch solution with a quasimomentum \( k \). The **real Fermi surface** \( F_{P, \mathbb{R}} \) is \( F_P \cap \mathbb{R}^n \).

Equivalently, \( k \in F_P \) means the existence of a non-zero solution of the equation \( P(k)v = 0 \). In the Euclidean case, we have \( u(x) = e^{ik \cdot x} p(x) \), where \( p(x) \) is a \( G \)-periodic function. Fermi surface plays in the periodic situation the role of the characteristic variety (the set of zeros of the symbol) of a constant coefficient differential operator.

Introducing a spectral parameter \( \lambda \), one arrives at the notion of the **Bloch variety**:

**Definition 7.** The (complex) **Bloch variety** \( B_P \) of the operator \( P \) consists of all pairs \( (k, \lambda) \in \mathbb{C}^{n+1} \) such that the equation \( Pu = \lambda u \) has a nonzero Bloch solution \( u \) with a quasimomentum \( k \). The **real Bloch variety** \( B_{P, \mathbb{R}} \) is \( B_P \cap \mathbb{R}^{n+1} \).

The Bloch variety \( B_P \) can be treated as the graph of a (multivalued) function \( \lambda(k) \), which is called the **dispersion relation**. If the spectra of the operators \( P(k) \) on \( M \) are discrete, we can single out continuous branches \( \lambda_j \) of this multivalued dispersion relation. They are called the **band functions** [75, 50]. The Fermi surface is obviously the zero level set of the dispersion relation.

In order to justify the notion of a band function, we need to guarantee discreteness of the spectrum of the operators \( P(k) \) on \( M \) for all \( k \in \mathbb{C}^n \). In other words, we need to exclude the pathological (but possible) situation of the spectrum of \( P \) covering the complex plane. One of the reasons for such strange spectral behavior is the Fredholm index being not equal to zero. However, even when the index is equal to zero, such pathology can occur, as the example of the operator \( e^{i\varphi} d/d\varphi \) on the unit circle shows. Self-adjointness of \( P \) is one of the conditions that would
obviously guarantee discreteness. Another example is a second-order periodic uniformly elliptic operator in $\mathbb{R}^n$ of the form

$$L = -\sum_{i,j=1}^{n} a_{ij}(x) \partial_i \partial_j + \sum_{i=1}^{n} b_i(x) \partial_i + c(x).$$  

(3.7)

More sufficient conditions can be found for example in [1].

**Lemma 8.** The Fermi and Bloch varieties are the sets of all zeros of entire functions of a finite order in $\mathbb{C}^n$ and $\mathbb{C}^{n+1}$, respectively.

This is proven in [50, Theorems 3.1.7 and 4.4.2] for the flat case. The case of a general abelian covering does not require any changes in the proof.

The lemma implies in particular, that the band functions $\lambda(k)$ are piecewise-analytic (e.g., when there is no level crossing, one has analyticity due to the standard perturbation theory [43]). This statement was originally proven in [83] for Schrödinger operators.

Another useful property of the Bloch and Floquet varieties is the relation between the corresponding varieties of the operators $P$ and $P^*$:

**Lemma 9.** [50, Theorem 3.1.5] A quasimomentum $k$ belongs to $F_{P^*}$ if and only if $-k \in F_P$. Analogously, $(k, \lambda) \in B_{P^*}$ if and only if $(-k, \lambda) \in B_P$. In other words, the dispersion relations $\lambda(k)$ and $\lambda^*(k)$ for the operators $P$ and $P^*$ are related as follows:

$$\lambda^*(k) = \lambda(-k).$$  

(3.8)

We will need to see how the structure of the functions of Floquet type (see Definition 3) and in particular, of Floquet solutions of our periodic equation reacts to the Floquet-Gelfand transform. For instance, in the constant coefficient case, where the role of the Floquet solutions is played by the exponential polynomials

$$e^{ik \cdot x} \sum_{|j| \leq N} p_j x^j,$$

such functions are Fourier transformed into distributions supported at the point $(-k)$. The next statement shows that under the Floquet-Gelfand transform, each Floquet type function (3.1) corresponds, in a similar way, to a (vector valued) distribution supported at the quasimomentum $(-k)$. This, and some other properties of Floquet solutions that play a crucial role in establishing the Liouville type theorems are collected in the next lemma.

Every Floquet type function $u$ with a real quasimomentum is of polynomial growth, and thus determines a (continuous linear) functional
on the previously defined space $C^0$ (see Theorem 1). If it satisfies the equation $Pu = 0$ for a periodic elliptic operator of order $m$, then as such a functional it is orthogonal to the range of the dual operator $P^* : C^m \to C^0$. According to Theorem 1, after the Floquet-Gelfand transform any such functional becomes a functional on $C^\infty(\mathbb{T}^n, \mathcal{E}^0)$ that is orthogonal to the range of the operator of multiplication by the Fredholm morphism $P^*(k) : \mathcal{E}^m \to \mathcal{E}^0$. The following auxiliary result (see [57]) describes all such functionals.

**Lemma 10.**

1. A continuous linear functional $u$ on $C^0$ is generated by a Floquet type function with a quasimomentum $k_0$ if and only if after the Floquet-Gelfand transform it corresponds to a functional on $C^\infty(\mathbb{T}^n, \mathcal{E}^0)$ which is a distribution $\phi$ that is supported at the point $-k_0$, i.e. has the form

$$
\langle \phi, f \rangle = \sum_{|j| \leq N} \left( q_j, \frac{\partial^{|j|} f}{\partial k^{|j|}} \right)_{L^2(M)},
$$

where $q_j \in L^2(M)$. The orders $N$ of the Floquet function and of the corresponding distribution $\phi$ are the same.

2. Let $a_k$ be the dimension of the kernel of the operator $P(k) : H^m_k \to L^2_k$. Then the dimension of the space of Floquet solutions of the equation $Pu = 0$ of order at most $N$ with a quasimomentum $k$ is finite and does not exceed $a_k q_{a,N}$.

The estimate on the dimension given in the Lemma 10 is very crude and can often be improved. The next Theorem 12 provides in some cases exact values of these dimensions. This theorem is the crucial part of the proof of the Liouville Theorem 17 of Section 4. In order to state and prove it, we need to introduce some notions first.

First of all, the analytic Hilbert bundle $\mathcal{E}^m$ is locally trivial for any $m$. Since the previous Lemma shows that our interest in Floquet solutions is local with respect to the quasimomentum $k$, we can trivialize the bundles and hence assume that the analytic families of operators $P(k)$ and $P^*(k)$ act in a fixed Hilbert space. At this moment, we will need the additional condition that the spectra of these operators are discrete (see the corresponding discussion earlier in the text). Assume now that zero is an eigenvalue of the adjoint operator $P^*(-k_0) : H^m_{-k_0} \to L^2_{-k_0}$ (since under the conditions we imposed on the operator, its Fredholm index is zero, this means that the operator $P(k_0)$ has eigenvalue zero as well). Suppose that algebraic and geometric multiplicities of $\lambda = 0$
coincide (this holds, for instance, in the self-adjoint case) and are equal to $r$. Let $\{e_j\}_{j=1}^r$ be a basis in the kernel of $P^*(-k_0)$. Consider a closed curve $\Upsilon$ in $\mathbb{C}$ separating 0 from the rest of the spectrum of $P^*(-k_0)$ and the corresponding (analytically depending on $k$ in a neighborhood of $k_0$) $r$-dimensional spectral projector $\Pi(k)$ for $P^*(-k)$. We will also need the following $r \times r$ matrix function

$$\lambda(k)_{ij} = \langle e_j, P^*(k)\Pi(k)e_i \rangle.$$  \hfill (3.10)

This matrix function is analytic with respect to $k$ in a neighborhood of $k_0$. Consider the Taylor expansion of $\lambda(k)$ around the point $k_0$ into homogeneous matrix-valued polynomials:

$$\lambda(k) = \sum_{l \geq 0} \lambda_l(k - k_0).$$  \hfill (3.11)

In this paper we will be mostly concerned with the first non-zero term $\lambda_0$ of the expansion.

We would like to mention that in what follows, the results will be invariant with respect to multiplication of the matrix $\lambda$ from both sides by invertible matrix-functions analytic in a neighborhood of $k_0$. This means, in particular, that if $r = 1$, then $\lambda(k)$ can equivalently be chosen to be equal to the analytic around $k_0$ branch of the eigenvalue of $P^*(-k)$ that vanishes at $k_0$.

**Definition 11.** Let $Q$ be a homogeneous polynomial in $n$ variables with matrix coefficients of dimension $r \times r$, and let $Q(D)$ be the differential matrix operator with the symbol $Q$. A $\mathbb{C}^r$-valued polynomial $p(x)$ in $\mathbb{R}^n$ is called $Q$-harmonic, if it satisfies the system of differential equations $Q(D)p = 0$.

Let $\mathcal{P}$ denote the vector space of all $\mathbb{C}^r$-valued polynomials in $n$ variables, and let $P_l$ be the subspace of all such homogeneous polynomials of degree $l$. So,

$$\mathcal{P} = \bigoplus_{l=0}^{\infty} P_l$$  \hfill (3.12)

and

$$\mathcal{P}_N := \bigoplus_{l=0}^{N} P_l$$  \hfill (3.13)

is the subspace of all such vector valued polynomials of degree at most $N$. If $Q(k)$ is a homogeneous polynomial of degree $s$ with values in $r \times r$ matrices, then one can define the matrix differential operator $Q(D)$ that in particular maps $P_{l+s}$ to $P_l$. If the determinant $\det Q$ is
not identically equal to zero, then this mapping is surjective for any \( l \) (this will follow from the proof of the theorem below). Hence, the mapping \( Q(D) : \mathcal{P} \to \mathcal{P} \) has a (non-uniquely defined) linear right inverse \( R \) that preserves the homogeneity of polynomials.

**Theorem 12.** Let zero be an isolated eigenvalue of multiplicity \( r \) of the operator \( P^*(-k_0) : H^m_{-k_0} \to L^2_{-k_0} \), and let \( \lambda(k) \) be defined in a neighborhood of \( k_0 \) as in (3.10)\(^{11}\). Let also \( \lambda_{l_0} \) be the first nonzero term of the Taylor expansion (3.11). Then

1. For any \( N \in \mathbb{N} \), the dimension of the space of Floquet solutions of the equation \( Pu = 0 \) in \( \mathbb{R}^n \) with the quasimomentum \( k_0 \) and of order at most \( N \) is finite and does not exceed \( rq_{n,N} \).

2. If \( \det \lambda_{l_0} \) is not identically equal to zero (for instance, this is the case when the eigenvalue is simple, i.e. \( r = 1 \)), then for any \( N \in \mathbb{N} \) the dimension of the space of Floquet solutions of the equation \( Pu = 0 \) in \( \mathbb{R}^n \) of order at most \( N \) and with the quasimomentum \( k_0 \) is equal to

\[
(3.14) \quad r \left( \begin{pmatrix} n + N \\ N \end{pmatrix} - \begin{pmatrix} n + N - l_0 \\ N - l_0 \end{pmatrix} \right).
\]

3. This dimension coincides with the dimension of the space of all \( \lambda_{l_0} \)-harmonic polynomials of degree of at most \( N \) with values in \( \mathbb{C}^r \). Moreover, given a linear right inverse \( R \) of the mapping \( \lambda_{l_0}(D) : \mathcal{P} \to \mathcal{P} \) that preserves homogeneity, one can construct an explicit isomorphism between the corresponding spaces.

**Proof.** As before, we will assume that the bundles are analytically trivialized around the point \( k_0 \) and hence all operators act between fixed spaces that we will denote \( H^m \) and \( H^0 \). In order to simplify notations, we assume that \( k_0 = 0 \) (this can always be achieved by a change of variables). Let us denote by \( N(k) \) the range of the projector \( \Pi(k) \) and choose a closed complementary subspace \( M \) to \( N(0) \) in \( H^m \). The subspace \( M \) stays complementary to \( N(k) \) in a neighborhood of 0 and so

\[
H^m = M \oplus N(k).
\]

Thus, \( P^*(-k) \) has zero kernel on \( M \) for all \( k \) in a neighborhood of 0. This implies that the range of \( P^*(-k) \) on \( M \) forms an analytic Banach vector bundle \( R(k) \) in a neighborhood of 0 (e.g., Theorem 1.6.13 of [50]).

\(^{11}\)Any analytic function in a neighborhood of \( k_0 \) that differs from \( \lambda(k) \) by a left and right multiplication by analytic invertible matrix functions will produce the same results in what follows.
Representing now the operator $P^*(-k)$ in the block form according to the decompositions

$$H^m = M \oplus N(k)$$

and

$$H^0 = R(k) \oplus N(k),$$

we get

$$P^*(-k) = \begin{pmatrix} B(k) & 0 \\ 0 & \tilde{\lambda}(-k) \end{pmatrix},$$

where $B(k)$ is an analytic invertible operator-function and the matrix analytic function $\tilde{\lambda}(-k)$ differs from $\lambda(-k)$ only by multiplying by an invertible analytic matrix function and thus for our purposes can be replaced by the latter one.

Let us now have a functional $\phi$ on $C^\infty(\mathbb{T}^n, \mathcal{E}_0)$ supported at 0, such that it is orthogonal to the range of the operator of multiplication by $P^*(k)$. Then it must be equal to zero on all sections of the bundle $R(k)$ (since they are all in the range, due to invertibility of $B(k)$). This means that the restriction of such functionals to the sections of the finite-dimensional bundle $N(k)$ is an one-to-one mapping. This reduces the problem to the following: find the dimension of the space of all distributions of order $N$ supported at the origin such that they are orthogonal to the sub-module generated by the matrix $\lambda(-k)$ in the module of germs of analytic vector valued functions. One can change variables to eliminate the minus sign in front of $k$. Due to the finiteness of the order of the distribution, the problem further reduces to the following: find the dimension of the cokernel of the mapping

$$\Lambda_N : \mathcal{P}_N \to \mathcal{P}_N.$$  

Here $\Lambda_N(p)$ for $p \in \mathcal{P}_N$ is the Taylor polynomial of order $N$ at 0 of the product $\lambda(k)p(k)$. Let us write the block matrix $\Lambda_{ij}$ of the operator $\Lambda_N$ that corresponds to the decomposition $\mathcal{P}_N = \bigoplus_{l=0}^N P_l$. Then $\Lambda_{ij} = 0$ for $i - j < l_0$. For $i - j \geq l_0$ the entry $\Lambda_{ij}$ is the operator of multiplication by $\lambda_{i-j}$ acting from $P_j$ into $P_i$. Since $\det(\lambda_{l_0})$ is not identically equal to zero, for $i - j = l_0$ the operator $\Lambda_{ij}$ of multiplication by $\lambda_{l_0}$ has zero kernel.

In order to prove the theorem, we need to find the dimension of the cokernel of $\Lambda_N$, as well as to obtain the cokernel’s description.

The first statement of the theorem is now obvious, since the dimension of the cokernel of $\Lambda_N$ cannot exceed the dimension of the ambient space $\mathcal{P}_N$, which is equal to $rq_{n,N}$. 

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Let us now approach the second statement. Since $\Lambda_N$ is a square matrix, $\dim \text{Coker} \Lambda_N = \dim \text{Ker} \Lambda_N$. The latter dimension, however, is easy to find, due to the triangular structure of the equation $\Lambda_N p = 0$, if it is written in the block matrix form according to the decomposition (3.13):

$$
\begin{pmatrix}
0 & \ldots & \lambda_{l_0} & \lambda_{l_0+1} & \ldots \\
0 & \ldots & 0 & \lambda_{l_0} & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & \lambda_{l_0} \\
0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
p_N \\
p_{N-1} \\
\vdots \\
p_1 \\
p_0 \\
\end{pmatrix} = 0.
$$

(3.15)

Here $p = p_0 + \ldots + p_N$ is the expansion of $p \in \mathcal{P}_N$ into homogeneous terms. Since $\det \lambda_{l_0} \neq 0$, one concludes immediately from (3.15) that $p_0 = \ldots = p_{N-l_0} = 0$, while other components are arbitrary. This gives the dimension of the kernel (and hence of the cokernel) of $\Lambda_N$ as

$$r \left( \begin{pmatrix} n + N \\ N \end{pmatrix} - \begin{pmatrix} n + N - l_0 \\ N - l_0 \end{pmatrix} \right),$$

which proves the second statement of the theorem.

In order to actually describe the elements of the cokernel of $\Lambda_N$ and hence prove the last assertion of the theorem, we need to find the kernel of the adjoint matrix $\Lambda_N^*$. The adjoint matrix acts in the space $\bigoplus_{l=0}^N P_l^*$, where $P_l^*$ can be naturally identified with the space of linear combinations with coefficients in $\mathbb{C}^r$ of the derivatives of order $l$ of the Dirac’s delta-function at the origin. Here we have $\Lambda_{ij}^* = 0$ for $j - i < l_0$, and for $j - i \geq l_0$ the entry $\Lambda_{ij}^*$ is the dual to the operator of multiplication by $\lambda_{j-i}$ acting from $P_i$ into $P_j$. In particular, since for $j - i = l_0$ the latter operator is injective, we conclude that the operators $\Lambda_{ij}^*$ are surjective. This enables one to describe the structure of the kernel of $\Lambda_N^*$ and hence of cokernel of $\Lambda_N$. Namely, let

$$\psi = (\psi_0, ..., \psi_N) \in \bigoplus_{l=0}^N P_l^*$$

be such that $\Lambda^* \psi = 0$. Due to the triangular structure of $\Lambda_N^*$, we can solve this system:

$$\sum_{j \geq i+l_0} \Lambda_{ij}^* \psi_j = 0, \quad i = 0, ..., N - l_0.$$
Taking the Fourier transform, we can rewrite this system in the form
\[ \sum_{j \geq i + l_0} \lambda_{j-i}(D) \hat{\psi}_j = 0, \ i = 0, \ldots, N - l_0, \]
where \( \hat{\psi} \) denotes the Fourier transform of \( \psi \). Therefore, \( \hat{\psi}_j \) is a homogeneous polynomial of degree \( j \) in \( \mathbb{R}^n \). For \( i = N - l_0 \) we have
\[ \lambda_{l_0}(D) \hat{\psi}_N = 0. \]
This equality means that \( \hat{\psi}_N \) can be chosen as an arbitrary \( \lambda_{l_0} \)-harmonic homogeneous polynomial of order \( N \). Moving to the previous equation, we analogously obtain
\[ \lambda_{l_0}(D) \hat{\psi}_{N-1} + \lambda_{l_0+1}(D) \hat{\psi}_N = 0, \]
or
\[ \lambda_{l_0}(D) \hat{\psi}_{N-1} = -\lambda_{l_0+1}(D) \hat{\psi}_N. \]
The right hand side is already determined, and the nonhomogeneous equation, as we concluded before, always has a solution, for instance
\[ -R \left( \lambda_{l_0+1}(D) \hat{\psi}_N \right). \]
This means that
\[ \hat{\psi}_{N-1} + R \left( \lambda_{l_0+1}(D) \hat{\psi}_N \right) \]
is a \( \lambda_{l_0} \)-harmonic homogeneous polynomial of order \( N - 1 \). We see that the solution \( \hat{\psi}_{N-1} \) exists and is determined up to an addition of any homogeneous \( \lambda_{l_0} \)-harmonic polynomial of degree \( N - 1 \). Continuing this process until we reach \( \hat{\psi}_0 \), we conclude that the mapping
\[ \psi = (\psi_0, \ldots, \psi_N) \rightarrow \phi = (\phi_0, \ldots, \phi_N), \]
where
\[ \phi_j = \hat{\psi}_j + R \sum_{i>j} \lambda_{i-j+l_0}(D) \hat{\psi}_i \]
establishes an isomorphism between the cokernel of the mapping \( \Lambda_N \) and the space of \( \lambda_{l_0} \)-harmonic polynomials of degree at most \( N \). This proves the theorem. \( \square \)

In the simplest possible cases, the theorem immediately implies the following:

**Corollary 13.** Under the hypotheses of Theorem 12 one has:

1. If \( k_0 \) is a noncritical point of a single band function \( \lambda(k) \) such that \( \lambda(0) = 0 \), then the dimension of the space of Floquet solutions of the equation \( Pu = 0 \) on \( X \) of order at most \( N \) with a
quasimomentum $k_0$ is equal to the dimension $q_{n-1,N}$ of the space of all polynomials of degree at most $N$ in $\mathbb{R}^{n-1}$.

(2) If the Taylor expansion at a point $k_0$ of a single band function $\lambda(k)$ such that $\lambda(0) = 0$ starts with a non-zero quadratic form\(^{12}\), then the dimension of the space of Floquet solutions of the equation $Pu = 0$ on $X$ of order at most $N$ with a quasimomentum $k_0$ is equal to the dimension $h_{n,N}$ of the space of harmonic (in the standard sense) polynomials of degree at most $N$ in $\mathbb{R}^n$. In particular, this condition is satisfied at nondegenerate extrema of dispersion curves (i.e., at non-degenerate spectral edges).

In both cases an isomorphism can be provided explicitly as in the previous theorem.

4. **Liouville type theorems for elliptic periodic systems**

After all the technology has been prepared, we can move on to establishing Liouville theorems for periodic equations. We will consider at the moment an arbitrary linear (square) matrix elliptic operator $P$ on the space $X$ of the abelian covering $X \to M$ with smooth $G$-periodic coefficients that satisfies the assumptions made in Section 2. As before, without loss of generality we can limit the consideration to the case $G = \mathbb{Z}^n$. Any of the standard meanings of ellipticity of a system would do, e.g. ellipticity in Petrovsky or Douglis-Nirenberg sense [28].

**Definition 14.** We say that the **Liouville theorem of order** $N \in \mathbb{N}$ **holds true for the operator** $P$, if the space $V_N(P)$ of solutions of the equation $Pu = 0$ on $X$ that for any compact subdomain $K \subset X$ can be estimated as

$$||u||_{L^2(gK)} \leq C_{K,u}(1 + |g|)^N$$

for all $g \in G$ is finite dimensional.

We say that the **Liouville theorem holds true for the operator** $P$, if it holds for any order $N \in \mathbb{N}$.

Abusing notations, we will call solutions from $V_0(P)$ **bounded solutions**.

Since obviously $V_{N_1}(P) \subset V_{N_2}(P)$ for $N_1 < N_2$, Liouville theorem of higher order implies the lower order ones. It is not clear **a priori** that the converse holds. The next results show that this is in fact true, at least in our situation of abelian coverings. This observation apparently has failed to be made in previous studies, which sometimes lead to

\(^{12}\) The form being not necessarily nondegenerate, which was required in [57].
investigations of some individual Liouville theorems, e.g. for $N = 0$ without noticing their simultaneous validity for all $N$.

We will start with an auxiliary statement, which is an analog of the classical theorem on the structure of distributions supported at a single point. It is a generalization of Lemma 25 in [57], where Fredholm rather than semi-Fredholm property is assumed. For the results of this section Lemma 25 in [57] would be sufficient, while we need the full strength of the lemma below to treat overdetermined systems and holomorphic functions further on in the paper.

**Lemma 15.** Let $T$ be a $C^\infty$-manifold and $P : T \to L(H_1, H_2)$ be a $C^\infty$-function with values in the space $L(H_1, H_2)$ of bounded linear operators between Hilbert spaces $H_1$ and $H_2$. Assume that for each $k \in T$ the operator $P(k)$ is right semi-Fredholm (e.g., [86]), i.e. it has a closed range and a finite dimensional cokernel. Then

1. If $P(k)$ is surjective for all points $k$ in $T$, then the multiplication operator

   $C^\infty(T, H_1) \xrightarrow{P(k)} C^\infty(T, H_2)$

   is surjective.

2. For any fixed $k_0 \in T$ the dimension of the space of functionals of the form

   $\left[ \sum_{j \leq N} D_{j,k}(<g_j(k), \phi>) \right]_{k_0}$

   that are orthogonal to the range of the operator (4.1) is finite. Here $g_j(k)$ are continuous linear functionals on $H_2$, $<g_j(k), \phi>$ denotes the duality between $H_2^*$ and $H_2$, $D_{j,k}$ are linear differential operators with respect to $k$ on $T$, and $N \in \mathbb{N}$.

3. If $P(k)$ is surjective for all points $k$ except of a finite subset $F \subset T$, then any continuous linear functional $g$ on the space of smooth vector functions $C^\infty(T, H_2)$ that annihilates the range of the multiplication operator

   $C^\infty(T, H_1) \xrightarrow{P(k)} C^\infty(T, H_2)$

   has the form

   $< g, \phi > = \sum_{k_l \in F} \left[ \sum_{j \leq N} D_{j,k_l}(<g_j(k_l), \phi>) \right]_{k_l}$

   in the notations of the previous statement of the lemma.
Proof: Let us establish the validity of the first statement of the lemma. Due to existence of partitions of unity, the statement is local. Locally, following for instance the proof of Theorem 2.7 in [86], one can construct a smooth one-sided (right) inverse $Q(k)$ to the operator function $P(k)$. Now multiplication by $Q(k)$ provides a right inverse to (4.1).

To prove the second statement, let us consider a closed subspace $M \subset H_1$ complementary to the kernel of $P(k_0)$. Then the operator $P(k_0) : M \to H_2$ is injective and Fredholm. This injectivity property is preserved in a neighborhood $U$ of $k_0$. In particular, the subspace $M(k) = P(k)(M) \subset H_2$ of finite codimension forms a smooth subbundle $\mathcal{M}$ in $U \times H_2 \to U$. Now, any smooth $H_2$-valued function $f(k)$ such that $f(k) \in M(k)$, belongs to the range of the operator (4.1). Hence, functionals orthogonal to the range can be pushed down to the smooth sections of the finite dimensional bundle $\mathcal{C} = \bigcup_{k \in U} H_2 / M(k)$ over $U$. It is clear that the functional one gets on this bundle preserves the structure (4.2). Such functionals on a finite dimensional bundle, however, form a finite dimensional space (for any fixed $N$).

The third statement can be proven analogously to the similar statement in [50, Corollary 1.7.2]. Namely, under the conditions of the statement, and taking into account the first claim of the lemma, any functional annihilating the range of the operator of multiplication by $P(k)$ must be supported at the finite set $F$ over which $P(k)$ is not surjective. We can reduce the consideration to a neighborhood $U$ of a single point $k_0 \in F$. Now, the proof of the second statement reduces the functional to one defined on smooth sections of a finite-dimensional bundle $\mathcal{C}$ over $U$, which is supported at the point $k_0$. Now the standard representation of distributions supported at a point implies (4.4).

The next theorem shows that existence of a polynomially growing solution implies existence of a non-zero bounded Bloch solution (i.e., a solution automorphic with respect to a unitary character of $G$).

**Theorem 16.** The equation $Pu = 0$ has a nonzero polynomially growing solution if and only if it has a nonzero Bloch solution with a real quasimomentum (such a solution is automatically bounded), i.e. if and only if the real Fermi surface $F_{P, \mathbb{R}} = F_P \cap \mathbb{R}^n$ is not empty.

Proof: Assume that $F = \emptyset$. Then $P(k)$ is surjective for all $k \in \mathbb{T}^n$. Indeed, if this were not the case, we could find a non-zero functional of the type (3.9) with $N = 0$ and some $k$. According to the first statement of Lemma 10, this would mean existence of a Bloch solution. Now, the first statement of Lemma 15 guarantees surjectivity of the mapping $C^\infty(\mathbb{T}^n, \mathcal{E}^m) \xrightarrow{P(k)} C^\infty(\mathbb{T}^n, \mathcal{E}^0)$. 


and hence the absence of any nontrivial functionals on $C^\infty(\mathbb{T}^n, \mathcal{E}^0)$ that annihilate the image of this mapping. Since under the Floquet-Gelfand transform $U$, any polynomially growing solution $u(x)$ of $Pu = 0$ is mapped to such a functional, we conclude that $u = 0$. □

The next result provides necessary and sufficient conditions under which Liouville theorems hold for equations of the type we consider. It also establishes that validity of Liouville type theorems does not depend on the order of polynomial growth.

**Theorem 17.** Under the conditions we have imposed on the covering and the operator $P$, the following statements are equivalent:

1. The number of points in the real Fermi surface $F_{P,R}$ is finite (equivalently, Bloch (or automorphic) solutions exist for only finitely many unitary characters $\gamma_k$).
2. The Liouville theorem holds for some $N \in \mathbb{N}$.
3. The Liouville theorem holds (i.e., it holds for any $N$).

**Proof:** Any Bloch solution with a real quasimomentum $k$ (i.e., corresponding to a unitary character) is bounded and hence belongs to the space $V_N(P)$ for any $N$. Since such solutions with different characters are linearly independent, validity of Liouville theorem for some value of $N$ implies that the number of the corresponding characters is finite. Since the characters of all Bloch solutions with real quasimomenta constitute the real Fermi variety, this proves the implication (2) $\Rightarrow$ (1).

The implication (3) $\Rightarrow$ (2) is obvious.

Let us now prove (1) $\Rightarrow$ (3). Let $u$ be a non-zero polynomially growing solution. It can be interpreted as a continuous functional on $C^0$ annihilating the range of the dual operator $P^* : C^m \to C^0$. After the Floquet-Gelfand transform $U$, one obtains a functional on $C^\infty(\mathbb{T}^n, \mathcal{E}^0)$ orthogonal to the range of the operator

$$C^\infty(\mathbb{T}^n, \mathcal{E}^m) \xrightarrow{P(k)} C^\infty(\mathbb{T}^n, \mathcal{E}^0).$$

By our assumption, $F$ is finite, thus the second statement of Lemma 15 and Theorem 12 finish the proof of statement (3). □

While this theorem establishes conditions under which Liouville theorem holds, it does not tell much about the dimensions of the spaces $V_N(P)$ of polynomially growing solutions, besides those being finite. It also does not address the structure of these solutions. The next result provides some estimates and even explicit formulas for the dimensions, as well a representation for the solutions.

**Theorem 18.** If the Liouville theorem holds for an elliptic operator $P$, then the following statements also hold, where $d_N = \dim V_N(P)$:
Each solution \( u \in V_N(P) \) can be represented as a finite sum of Floquet solutions:

\[
\sum_{k \in F_{P,R}} \sum_j u_{k,j}(x),
\]

where each \( u_{k,j} \) is a Floquet solution with a quasimomentum \( k \), and \( F_{P,R} = F_P \cap \mathbb{R}^n \).

(2) For all \( N \in \mathbb{N} \), we have

\[
d_N \leq d_0 q_{n,N} < \infty,
\]

where \( q_{n,N} \) is the dimension of the space of all polynomials of degree at most \( N \) in \( n \) variables.

(3) Assume that the spectra of operators \( P(k) \) are discrete for all \( k \) and that for each real quasimomentum \( k \in F_{P,R} \) the conditions of Theorem 12 are satisfied. Then for each \( N \in \mathbb{N} \) the dimension \( d_N \) of the space \( V_N(P) \) is equal to

\[
\sum_{k \in F_{P,R}} r_k \left( \left( \begin{array}{c} n + N \\ N \end{array} \right) - \left( \begin{array}{c} n + N - l_0(k) \\ N - l_0(k) \end{array} \right) \right).
\]

Here \( r_k \) and \( l_0(k) \) are respectively the multiplicity of the zero eigenvalue and the order of the first non-zero Taylor term of the dispersion relation at the point \( k \in F_{P,R} \) (see these notions explained before Theorem 12). The terms in this sum are the dimensions of the spaces of \( \chi_{l_0(k)}(D) \)-harmonic polynomials, and polynomially growing solutions can be described in terms of these polynomials analogously to Theorem 12.

**Proof:** All these statements follow immediately from Lemma 10, Theorem 12, Lemma 15, and Theorem 17.

5. Examples of Liouville theorems for specific operators

Let us recall that the \( L^2 \)-spectrum of the operator \( P \) is the union over \( k \in B \) of the spectra of \( P(k) \) (e.g., [50, Theorem 4.5.1] and [75, Theorem XIII.85]). In other words, the spectrum of \( P \) coincides with the range of the dispersion relation over the Brillouin zone \( B \). We have also discussed that the real Fermi surface for \( P \) is just the zero level set for the dispersion relation over the Brillouin zone. Theorem 17 of the previous section shows that the Liouville theorem holds if and only if this Fermi surface is finite\(^{13} \). One expects this to happen

\(^{13}\)In particular, outside the spectrum the Liouville theorem holds vacuously, according to Theorem 17 and Theorem 5.5.1 in [50].
normally at extrema of the dispersion relation (albeit this is neither necessary, nor sufficient). In other words, speaking for instance about the selfadjoint case, one should expect Liouville theorem to hold mainly when zero is at the edge of a spectral gap, although it is possible in principle to have interior points of the spectrum where such a thing could occur as well\(^{14}\). One notices that the cases considered in [5, 60, 67] all correspond to zero being at the bottom of the spectrum (in the non-selfadjoint case we mean by the bottom of the spectrum the generalized principal eigenvalue, see [2, 57, 62] and Section 5.4 below). This explains why the homogenization techniques employed in these papers could be successful. Indeed, homogenization works exactly at the bottom of the spectrum. The results of this work show that one should also consider internal spectral edges, where the standard homogenization does not apply.

So, let us now try to look at some examples where one can apply the results of the previous sections. Theorem 17 is applicable to any elliptic periodic equation or system of equations on any abelian covering of a compact manifold. According to this theorem, one only needs to establish the finiteness of the real Fermi surface. On the other hand, Theorem 18, which provides formulas for the dimensions of the spaces of polynomially growing solutions, as well as some representations of such solutions, is more demanding. It requires that one guarantees discreteness of the spectra of the ‘cell’ operators \(P(k)\), and most of all, requires understanding of the analytic structure of the dispersion curve near the Fermi surface. Concerning this structure, it is expected that the following is true:

**Conjecture 19.** Let \(P\) be a ‘generic’ self-adjoint second-order elliptic operator with periodic coefficients and \((\lambda_-, \lambda_+)\) be a nontrivial gap in its spectrum. Then each of the gap’s endpoints is a unique (modulo the dual lattice) and non-degenerate extremum of a single band function \(\lambda_j(k)\).

This conjecture is crucial in many problems of mathematical physics, spectral theory, and homogenization (see its further discussion in Section 8) and is widely believed to hold. Unfortunately, the only known theorem of this kind is the recent result of [45], which states that generically a gap edge is an extremum of a single band function. Theorem 17 shows that validity of this conjecture would imply the following statement:

\(^{14}\)Two spectral zones with touching edges could provide such an example.
Conjecture 20. ‘Generically’, at the spectral edges of periodic elliptic operators in \( \mathbb{R}^n \), the Liouville theorem holds and the dimension of the space \( V_N \) is equal to the dimension \( h_{n,N} \) of the space of all harmonic polynomials of order at most \( N \) in \( n \) variables.

A similar conjecture probably holds for equations on abelian coverings of compact manifolds, while there one expects the rank of the deck group enter the answer rather than the dimension of the manifold (see Section 8).

At the bottom of the spectrum, however, much more is known [15, 16, 18, 31, 44, 74, 78]. Even this limited (in terms of spectral location) information together with Theorems 17 and 18 provides one with many specific examples that go far beyond the equations considered in [5, 60, 67].

The first trivial remark is that if zero is outside the spectrum of the operator \( P \), then according to Theorem 17 the Liouville property holds vacuously. Indeed, in this case the real Fermi surface is empty, and hence equation \( Pu = 0 \) has no polynomially growing solutions. Let us look now at some less trivial examples.

5.1. Schrödinger operators. Let \( X \to M \) be, as before, a non-compact abelian covering of a \( d \)-dimensional compact manifold and \( H = -\Delta + V(x) \) be a Schrödinger operator on \( X \) with a periodic real valued potential \( V \in L^{r/2}_{loc}(X), r > d \). Then the result of [44] for \( X = \mathbb{R}^d \) and of [47] in the general case states that the lowest band function \( \lambda_1(k) \) has a unique non-degenerate minimum \( \Lambda_0 \) at \( k = 0 \). All other band functions are strictly greater than \( \Lambda_0 \). In particular, the bottom of the spectrum of the operator \( H \) is at \( \Lambda_0 \). Let us assume that \( \Lambda_0 = 0 \) (or replace the operator with \( H = -\Delta + V(x) - \Lambda_0 \)). Then Theorems 17 and 18 become applicable, since the real Fermi surface consists of a single point \( k = 0 \) and the band function has a simple non-degenerate minimum at this point (i.e., \( r = 1 \) and \( l_0 = 2 \) in notations of Theorem 18). Thus, one obtains

**Theorem 21.** (1) The Liouville theorem holds for the operator \( H - \Lambda_0 \).

(2) The dimension of the space \( V_N(H - \Lambda_0) \) equals \( h_{n,N} \) (where \( n \) is the free rank of \( G \)).

(3) Every solution \( u \in V_N(H - \Lambda_0) \) of the equation \( Hu - \Lambda_0 u = 0 \) is a Floquet solution of order \( N \) with quasimomentum \( k = 0 \), i.e. it can be represented as

\[
  u(x) = \sum_{|j| \leq N} [x]^j p_j(x)
\]
with $G$-periodic functions $p_j$.

5.2. Magnetic Schrödinger operators. Consider the self-adjoint magnetic Schrödinger operator in $\mathbb{R}^n$

\begin{equation}
H = (i\nabla + A)^2 + V
\end{equation}

with periodic electric and magnetic potentials $V$ and $A$, respectively.\(^{15}\) Introduction of a periodic magnetic potential into the operator is known to change properties of the Fermi surface significantly (e.g., [31, 78]). However, small magnetic potentials do not destroy the properties of our current interest. Indeed, let $V$ be like in the previous statement, then there exists $\epsilon > 0$ (depending on $V$) such that for any periodic real valued magnetic potential $A$ such that

\begin{equation}
||A||_{L^r(\mathbb{T}^n)} < \epsilon
\end{equation}

and

\begin{equation}
\int_{\mathbb{T}^n} A(x)dx = 0
\end{equation}

the following holds true: The lowest band function $\lambda_1(k)$ of $H$ attains a unique non-degenerate minimum $\Lambda_0$ at a point $k_0$ (albeit, not necessarily at $k = 0$). All other band functions are strictly greater than $\Lambda_0$. Indeed, when both $V$ and $A$ are sufficiently small, this is proven in [31]. It is not hard, though, to allow for arbitrary electric and small magnetic potential. Indeed, the case when $A = 0$ is covered by the result of [44] (see the previous subsection). Now, the statement of Lemma 8 (see also [50, Theorem 4.4.2]) can be easily extended without any change in the proof to include analyticity with respect to the potentials (e.g., [31]). Namely, there exists an entire function $f(k, \lambda, A, V)$ of all its arguments such that $f(k, \lambda, A, V) = 0$ is equivalent to

\begin{equation}
(k, \lambda) \in B_{(i\nabla + A)^2 + V},
\end{equation}

where as before $B_H$ is the Bloch variety of the operator $H$. This, together with the just mentioned result of [44] for $A = 0$, imply that the stated feature of the lowest band function still holds for sufficiently small magnetic potentials.\(^{16}\)

Now one uses Theorems 17 and 18 again to obtain

\(^{15}\)One can see that the considerations and the result of this section immediately extend to more general abelian coverings. We avoid doing so, in order not to introduce any new notions required for defining the magnetic operators in such a setting.

\(^{16}\)The degree of smallness of $A$ depends on the electric potential $V$. 
Theorem 22. Let $V$, a sufficiently small $A$, and $k_0$ are as described above, then

1. The Liouville theorem holds for $u \in V_N(H - \Lambda_{k_0})$.
2. Any solution $u \in V_N(H - \Lambda_{k_0})$ is representable in the Floquet form
   
   $$v(x) = e^{ik_0 \cdot x} \sum_{|j| \leq N} x^j p_j(x)$$

   with periodic functions $p_j(x)$.
3. The dimension of the space $V_N(H - \Lambda_{k_0})$ is equal to $h_{n,N}$.

Note that the normalization (5.2) can be always achieved by a gauge transformation, which does not affect the spectrum and the Liouville property.

Besides magnetic Schrödinger operators with small magnetic potentials, another very special subclass is formed by the so called Pauli operators. Consider the following Pauli operators in $\mathbb{R}^2$:

$$P_\pm = (i\nabla + A)^2 \pm (\partial_{x_1} A_2 - \partial_{x_2} A_1),$$

where $A(x) = (A_1(x), A_2(x))$. The structure of the dispersion curves at the bottom of the spectrum of such an operator was studied in [15]. It was shown that the dispersion relation for $P_\pm$ attains at the point $k = 0$ its single non-degenerate minimum with the value $\Lambda_0 = 0$. This implies that the result for the Pauli operators holds exactly like for the Schrödinger operator in the previous subsection, and in fact in a more precise form, since the minimal value $\Lambda_0$ of the band function is known. Namely, every solution $u \in V_N(P_\pm)$ of the equation $P_\pm u = 0$ is representable in the form (1.4). The dimension of the space $V_N(P_\pm)$ is equal to the dimension $h_{2,N} = 2N + 1$ of the space of harmonic polynomials of degree up to $N$ in two variables.

The cases above of small magnetic potentials and of Pauli operators form a rather special subclass of periodic magnetic Schrödinger operators, in the sense that one still finds a single non-degenerate minimum of band functions at the bottom of the spectrum. This, however, is not true anymore for the whole class of periodic magnetic Schrödinger operators, which should influence significantly the Liouville theorems. Although there is not much known here yet, the results of [78] provide the first glimpse at the possibilities. In that paper the following operator in $\mathbb{R}^2$ is considered:

$$M_t = (i\nabla + tA(x))^2 - t^2,$$

where the magnetic potential $A(x)$ is $(0, a(x_1))$ with the 1-periodic function $a(x_1)$ that is equal to $\text{sign}(x_1)$ on $[-0.5, 0.5]$ and $t \in \mathbb{R}$. The
main result of \[78\] is that for \(|t| < 2\sqrt{3}\) the bottom of the spectrum of \(M_t\) is 0 and it is attained at the quasimomentum \(k = 0\), where the band function has a simple non-degenerate minimum. This implies immediately the same Liouville theorem as for the Pauli operators. The situation changes for \(|t| = 2\sqrt{3}\), when this minimum (which is still 0 and attained at \(k = 0\)) becomes degenerate. Namely, it is shown in \[78\] that the lowest eigenvalue \(\Lambda_1(k)\) of the operators \(M_t(k)\) for \(|t| = 2\sqrt{3}\) has the following Taylor expansion at \(k = 0\):

\[
\Lambda_1(k) = k_1^2 + \frac{1}{42} k_2^4 - \frac{1}{10} k_1^2 k_2^2 + O(|k|^6).
\]

This shows the degeneration of the Hessian. On the other hand, since the first nonzero homogeneous term of the expansion is of order two, Theorem 18 still says that the Liouville theorem sounds the same as before (i.e., the dimensions are equal to \(h_{2,N}\)). This changes, though, for a little bit larger values of \(t\), when two points of extremum appear and the bottom of the spectrum shifts to the negative half-axis. This implies that the Liouville theorem will apply after shifting to the bottom of the spectrum (i.e., after adding an appropriate positive scalar term to \(M_t\)), but the dimensions of the spaces \(V_N\) of polynomially growing solutions will increase above \(h_{2,N}\).

**Theorem 23.**

1. The Liouville theorem holds for the operators \(P_\pm\) and \(M_t\) for \(|t| \leq 2\sqrt{3}\).
2. The dimensions of the corresponding spaces \(V_N\) are equal to \(h_{2,N}\).
3. Every solution \(u \in V_N\) can be represented as

\[
u(x) = \sum_{|j| \leq N} x^j p_j(x)
\]

with \(G\)-periodic functions \(p_j\).
4. For a sufficiently small \(\varepsilon > 0\) and the values \(|t| \in (2\sqrt{3}, 2\sqrt{3} + \varepsilon)\), the Liouville theorem holds at the bottom of the spectrum of the operator \(M_t\), while the dimension of the space \(V_N(M_t)\) is equal to \(2h_{2,N}\).

It is also conjectured in \[78\] that for some values of \(t\), an operator \(M_t\) with the magnetic potential equal to \(A(x) = (\text{sign}(x_2), \text{sign}(x_1))\) on the cube \([-0.5, 0.5]^2\) and \(\mathbb{Z}^2\)-periodic, will exhibit a complete degeneration of the bottom of the spectrum in the sense of complete vanishing of the Hessian. If this happens to be true, it would also lead, according to Theorem 18, to increase in the dimensions of the spaces \(V_N\).
5.3. **Operators admitting regular factorization.** A very general and wide class of selfadjoint second-order periodic operators $H$ on $\mathbb{R}^n$, for which one can study thoroughly the dispersion curves at the bottom of the spectrum, was introduced in [16, 17]. It consists of operators that allow for what the authors of [16, 17] call *regular factorization*:

$$H = f(x)b(D)^*G(x)b(D)f(x).$$

(5.4)

Here $b(D) = \sum b_j D_{x_j}$ is a constant coefficients first-order linear differential operator, the periodic function $f(x)$ is such that $f, f^{-1} \in L^\infty(\mathbb{R}^n)$, and the periodic matrix-function $G(x)$ satisfies $c_0 I \leq G(x) \leq c_1 I$ for some $0 < c_0 \leq c_1$. Then, it was shown in [17] that the bottom of the spectrum of $H$ is 0 and it is attained at the quasimomentum $k = 0$, where the band function has a simple non-degenerate minimum. As in the previous subsections, this implies

**Theorem 24.**

(1) The Liouville theorem holds for the operator $H$ (5.4).

(2) $\dim V_N(H) = h_{n,N}$.

(3) Every solution $u \in V_N(H)$ of the equation $Hu = 0$ can be represented as

$$u(x) = \sum_{|j| \leq N} x^j p_j(x)$$

with $G$-periodic functions $p_j$.

5.4. **Non-selfadjoint second-order operators.** All examples discussed so far in this section addressed self-adjoint operators only. There is, however, a class of (in general non-selfadjoint) second-order operators in $\mathbb{R}^n$, which can be studied thoroughly and which plays an important role in probability theory.

Consider second-order uniformly elliptic operators of the following form (3.7):

$$L = -\sum_{i,j=1}^n a_{ij}(x) \partial_i \partial_j + \sum_{i=1}^n b_i(x) \partial_i + c(x)$$

(5.5)

with coefficients that are real and periodic.

For an operator of this type, an important function $\Lambda(\xi) : \mathbb{R}^n \to \mathbb{R}$ can be introduced, whose properties were studied in detail in [2, 62, 74]. It is defined by the condition that the equation

$$Lu = \Lambda(\xi)u$$

has a *positive* Bloch solution of the form

$$u_{\xi}(x) = e^{\xi \cdot x} p_\xi(x),$$

(5.6)
where \( p_\xi(x) \) is \( G \)-periodic.

We also need to define the following number:

\[
\Lambda_0 = \max_{\xi \in \mathbb{R}^n} \Lambda(\xi).
\]  

It follows from [2, 62] that \( \Lambda_0 \) can also be described as follows:

\[
\Lambda_0 = \sup \{ \lambda \in \mathbb{R} \mid \exists u > 0 \text{ such that } (L - \lambda)u = 0 \text{ in } \mathbb{R}^n \}.
\]  

In the self-adjoint case, \( \Lambda_0 \) is the bottom of the spectrum of the operator \( L \). The common name for \( \Lambda_0 \) is the generalized principal eigenvalue of the operator \( L \) in \( \mathbb{R}^n \).

We assemble the information we need about the function \( \Lambda(\xi) \) and \( \Lambda_0 \) in the following lemma. The reader can find proofs of these statements in [2, 62, 74] and more detailed references in [57].

**Lemma 25.**  
(1) The value \( \Lambda(\xi) \) is uniquely determined for any \( \xi \in \mathbb{R}^n \).

(2) The function \( \Lambda(\xi) \) is bounded from above, strictly concave, analytic, and has a nonzero gradient at all points except at its maximum point.

(3) Consider the operator

\[
L(\xi) = e^{-\xi \cdot x}Le^{\xi \cdot x} = L(x, D - i\xi)
\]

on the torus \( \mathbb{T}^n \). Then \( \Lambda(\xi) \) is the principal eigenvalue of \( L(\xi) \) with a positive eigenfunction \( p_\xi \). Moreover, \( \Lambda(\xi) \) is algebraically simple.

(4) The Hessian of \( \Lambda(\xi) \) is non-degenerate at all points.

(5) \( \Lambda_0 \geq 0 \) if and only if the operator \( L \) admits a positive (super)solution. This condition is satisfied in particular when \( c(x) \geq 0 \).

(6) \( \Lambda_0 \geq 0 \) if and only if the operator \( L \) admits a positive solution of the form (5.6).

(7) \( \Lambda_0 = 0 \) if and only if the equation \( Lu = 0 \) admits exactly one normalized positive solution in \( \mathbb{R}^n \).

(8) If \( c(x) = 0 \), then \( \Lambda_0 = 0 \) if and only if \( \int_{\mathbb{T}^n} b(x)\psi(x) \, dx = 0 \),

where \( \psi \) is the principal eigenfunction of \( L^* \) on \( \mathbb{T}^n \), and \( b(x) = (b_1(x), \ldots, b_n(x)) \). In particular, divergence form operators satisfy this condition.

(9) Let \( \xi \in \mathbb{R}^n \), and assume that \( u_\xi(x) = e^{\xi \cdot x}p_\xi(x) \) and \( u^*_\xi \) are positive Bloch solutions of the equations \( Lu = 0 \) and \( L^*u = 0 \), respectively. Denote by \( \psi \) the periodic function \( u_\xi u^*_\xi \). Consider
the function

$$\tilde{b}_i(x) = b_i(x) - 2 \sum_{j=1}^{n} a_{ij}(x)\{\xi_j + (p_\xi(x))^{-1} \partial_j p_\xi(x)\}.$$  

Then $\Lambda_0 = 0$ if and only if

$$\int_{\mathbb{T}^n} \tilde{b}_j(x) \psi(x) \, dx = 0, \ j = 1...n.$$  

Now one can describe Liouville type theorems for an operator of the form (5.5) assuming that $\Lambda_0 \geq 0$. This assumption implies that the operator admits a positive periodic supersolution.

It is shown in [57] that if $\Lambda(0) \geq 0$, the Fermi surface $F_L$ can touch the real space only at the origin (modulo the reciprocal lattice $G^*$) and in this case $\Lambda(0) = 0$.

In fact, we have the following stronger result which extends Lemma 15 in [57] to non-real $\lambda$. In addition, the proof of the statement below is more elementary than in that lemma. We remind the reader that $G^* \subset \mathbb{C}^n$ below denotes the dual lattice to $G$.

**Lemma 26.** Let $k = \gamma - i\xi \in \mathbb{C}^n$. If $\text{Re} \lambda < \Lambda(\xi)$, then $(k, \lambda)$ does not belong to the Bloch variety $B_L$ of the operator $L$. Moreover, if $\text{Re} \lambda = \Lambda(\xi)$, then $(k, \lambda) \in B_L$ if and only if $\gamma \in G^*$, $\xi$ belongs to the zero level set $\Xi$ of $\Lambda(\xi)$, and $\text{Im} \lambda = 0$.

**Proof:** Without loss of generality, we may assume that $\xi = 0$, $\Lambda(\xi) = 0$, and $L1 = 0$. Assume that $(k, \lambda) \in B_L$, and let $u(x) = e^{i\gamma \cdot x}p(x)$ be a Floquet solution with a quasimomentum $k$ of the equation

$$Lu = \lambda u \quad \text{in} \ \mathbb{R}^n,$$

where $\text{Re} \lambda \leq 0$. Take complex conjugate:

$$L\bar{u} = \bar{\lambda}\bar{u} \quad \text{in} \ \mathbb{R}^n.$$  

Next, we compute

$$L(|u|^2) = \bar{u}Lu + uL\bar{u} - 2 \sum_{i,j=1}^{n} a_{ij}u_{x_i}\bar{u}_{x_j} = 2\text{Re} \lambda |u|^2 - 2 \sum_{i,j=1}^{n} a_{ij}u_{x_i}\bar{u}_{x_j}.$$  

Notice that for each $\zeta \in \mathbb{C}^n$, we have

$$\sum_{i,j=1}^{n} a_{ij}\zeta_i\bar{\zeta}_j = \sum_{i,j=1}^{n} a_{ij}\text{Re} \zeta_i\text{Re} \zeta_j + \sum_{i,j=1}^{n} a_{ij}\text{Im} \zeta_i\text{Im} \zeta_j \geq 0.$$  

Therefore,

(5.9)  

$$L(|u|^2) \leq 0.$$
Thus, \(|u|^2 = |p(x)|^2\) is a periodic nonnegative subsolution of the equation \(Lu = 0\) in \(\mathbb{R}^n\). By the strong maximum principle \(|u(x)|^2 = \text{constant}\). In particular, \(L(|u|^2) = 0\). Since we have equality in (5.9), it follows that \(\Re \lambda = 0\) and
\[
\sum_{i,j=1}^{n} a_{ij} u_{x_i} \bar{u}_{x_j} = \sum_{i,j=1}^{n} a_{ij} \Re u_{x_i} \Re u_{x_j} + \sum_{i,j=1}^{n} a_{ij} \Im u_{x_i} \Im u_{x_j} = 0.
\]
It follows that \(u = \text{constant}\) and therefore, \(e^{i\gamma \cdot x}\) is a periodic function. Consequently, \(\gamma \in \Gamma^*\). Since \(L1 = 0\), it follows that \(\Im \lambda = 0\). □

Theorem 17 now shows that if \(\Lambda(0) > 0\), the Liouville Theorem holds vacuously and the equation \(Lu = 0\) does not admit any nontrivial polynomially growing solution.

Suppose now that \(\Lambda(0) = 0\). It follows from the Lemma 25 that if \(\Lambda_0 > 0\), then the point \(k = 0\) is a non-critical point of the dispersion relation, and if \(\Lambda_0 = 0\) then \(k = 0\) is a non-degenerate extremum. In terms of Theorem 18, in the first case \(l_0 = 1\), while in the second \(l_0 = 2\), while in both cases \(r = 1\). Now Theorems 17 and 18 imply that if \(\Lambda(0) = 0\) (i.e. \(0 \in F_{L,\mathbb{R}}\)), then the Liouville theorem holds, and every solution \(u \in V_N(L)\) is representable in the form (1.4). The dimension of the space \(V_N(L)\) is equal to \(h_{n,N}\) in the case when \(\Lambda_0 = 0\), and to \(q_{n-1,N}\) when \(\Lambda_0 > 0\).

We summarize these results in the following statement:

**Theorem 27.** Let \(L\) be a periodic operator of the form (5.5) such that \(\Lambda_0 \geq 0\). Then

1. The Liouville theorem holds vacuously if \(\Lambda(0) > 0\), i.e. the equation \(Lu = 0\) does not admit any nontrivial polynomially growing solution.
2. If \(\Lambda(0) = 0\) and \(\Lambda_0 > 0\), the Liouville theorem holds for \(L\),
\[
\dim V_N(L) = q_{n-1,N} = \left( \frac{n + N - 1}{N} \right),
\]
and every solution \(u \in V_N(L)\) of the equation \(Lu = 0\) can be represented as
\[
u(x) = \sum_{|j| \leq N} x^j p_j(x)
\]
with \(G\)-periodic functions \(p_j\).
3. If \(\Lambda(0) = 0\) and \(\Lambda_0 = 0\), the Liouville theorem holds for \(L\),
\[
\dim V_N(L) = h_{n,N} = \left( \frac{n + N}{N} \right) - \left( \frac{n + N - 2}{N - 2} \right),
\]
and every solution \( u \in V_N(L) \) of the equation \( Lu = 0 \) can still be represented as

\[
u(x) = \sum_{|j| \leq N} x^j p_j(x)\]

with \( G \)-periodic functions \( p_j \).

It is interesting to notice that for self-adjoint operators, at the spectral edges one never faces the case \( l = 1 \), as it does happen above when \( \Lambda(0) = 0 \) and \( \Lambda_0 > 0 \).

In this section, we considered the flat case \( X = \mathbb{R}^n \) only. However, the result still holds on general abelian coverings, which does not require any significant change in the proof.

6. Liouville Theorems for Overdetermined Elliptic Periodic Systems and for Holomorphic Functions

We will show here how the techniques and results of the preceding sections can be applied to some overdetermined elliptic systems, in particular to the Cauchy-Riemann operators on abelian covers of compact complex manifolds. In fact, the main construction stays essentially the same, so we will be brief in its description.

We consider as before an abelian covering \( X \overset{\psi}{\to} M \) of a compact Riemannian manifold. Let also \( L_j \ (j = 0, 1, \ldots) \) be finite dimensional smooth vector bundles on \( M \) equipped with a Hermitian metric and \( \mathcal{L}_j \) be the sheaves of their smooth sections. Suppose that we have an elliptic complex of differential operators

\[
0 \to \mathcal{L}_0 \overset{P}{\to} \mathcal{L}_1 \overset{P_1}{\to} \mathcal{L}_2 \overset{P_2}{\to} \ldots
\]

We can lift this complex to an elliptic complex on \( X \) for which we will use the same notations:

\[
0 \to \mathcal{L}_0 \overset{P}{\to} \mathcal{L}_1 \overset{P_1}{\to} \mathcal{L}_2 \overset{P_2}{\to} \ldots
\]

Notice that the lifted bundles have a natural \( G \)-action with respect to which the operators are periodic.

We will be interested in the spaces of the global solutions \( u(x) \) of the equation \( Pu = 0 \) on \( X \) that have polynomial growth in the sense that for any compact \( K \in X \)

\[
\|u\|_{L^2(gK, L_0)} \leq C(1 + |g|)^N
\]

for some \( C, N \), and every \( g \in G \). As before, without loss of generality we can assume that \( G = \mathbb{Z}^n \). The Floquet-Gelfand transform
reduces the elliptic complex (6.1) and its dual to the direct integrals with respect to the quasimomentum $k$ of elliptic complexes on $M$

\begin{equation}
(6.3) \quad 0 \leftrightarrow \mathcal{E}^1 \xrightarrow{P(k)} \mathcal{E}^2 \xleftarrow{P_1(k)} \ldots
\end{equation}

and

\begin{equation}
(6.4) \quad 0 \leftarrow \mathcal{F}^1 \xleftarrow{P^*(k)} \mathcal{F}^2 \xrightarrow{P_1^*(k)} \ldots,
\end{equation}

where $\mathcal{E}^j, \mathcal{F}^j$ are appropriately defined analytic Banach vector bundles over $\mathbb{C}^n$. As previously, solutions of polynomial growth of $Pu = 0$ after Floquet-Gelfand transform become functionals on $C^\infty(\mathbb{T}^*; \mathcal{E}^l)$ orthogonal to the range of the operator

\begin{equation}
(6.5) \quad C^\infty(\mathbb{T}^*; \mathcal{F}^2) \xrightarrow{P^*(k)} C^\infty(\mathbb{T}^*; \mathcal{F}^1).
\end{equation}

Notice that due to the ellipticity of the complex and compactness of the base, it is Fredholm as a complex of operators between the appropriate Sobolev spaces (e.g., [40, Vol. III, Sections 19.4 and 19.5], [76, Section 3.2., in particular Theorem 13], [82, Section IV.5]). Hence, the operators $P^*(k)$ on $M$ belong to the set $\Phi_r$ of the right semi-Fredholm operators between the corresponding Sobolev spaces of sections. This means that they have closed ranges of finite co-dimension, while having infinite dimensional kernels. It is also clear that as in the elliptic case before, $P(k)$ depends analytically (in fact, polynomially) on $k$.

The Fermi surface $F_P$ of $P$ is introduced as the set of all values of $k$ for which the equation $P(k)u = 0$ has a nonzero solution\footnote{It can be shown similarly to the elliptic case that the Fermi surface, as in the elliptic case, is an analytic subset of $\mathbb{C}^n$.}. It is clear that it coincides with the set where the dual operator $P^*(k)$ has a nonzero cokernel. This enables us to carry over the considerations of the previous section to prove the following result:

**Theorem 28.** The following conditions are equivalent:

1. The (real) Fermi surface $F_{P,\mathbb{R}}$ is finite.
2. For some $N \in \mathbb{N}$ the dimension of the space $V_N(P)$ of solutions of polynomial growth of order $N$ is finite (i.e., the Liouville theorem of order $N$ holds).
3. The dimension of $V_N(P)$ is finite for any $N \in \mathbb{N}$ (i.e., the Liouville theorem holds).

If any (and hence all) of these conditions holds, then any solution of polynomial growth is a linear combination of Floquet solutions.
Proof: Since any point \( k \in F_{P,R} \) provides a bounded Bloch solution and those for different values of \( k \) (modulo the dual lattice) are linearly independent, the implications \((3) \Rightarrow (2) \Rightarrow (1)\) are obvious. Let us now assume \((1)\) and prove \((3)\). According to Lemma 15, all functionals of order \( N \) orthogonal to the range of \((6.5)\) can be expressed as linear combinations over \( k \in F_{P,R} \) of the form \((4.4)\). The same Lemma together with Lemma 10 and finiteness of \( F_{P,R} \) show the finite dimensionality of this space and the Floquet representation for all solutions of this type.

There is the following interesting case when this theorem can be applied.

Theorem 29. Let \( X \xrightarrow{G} M \) be an abelian cover of a compact (complex) analytic manifold \( M \). Let \( X \) be equipped with the lift of an arbitrary Riemannian metric from \( M \) and \( \rho \) be the corresponding distance. Let us also denote by \( A_N(X) \) the space of all analytic functions on \( X \) growing not faster than \( C(1 + \rho(x))^N \). Then:

1. The real Fermi surface of the \( \bar{\partial} \)-operator defined on functions on \( X \) contains only the origin: \( F_{\bar{\partial},R} = \{0\} \).
2. \( A_0(X) \) consists of constants.
3. For any \( N \), \( \dim A_N(X) < \infty \). All elements of the space \( A_N(X) \) are holomorphic Floquet-Bloch functions with quasimomentum \( k = 0 \).
4. There exists a finite family of holomorphic functions \( \{f_j\} \) on \( X \) such that all elements of \( A_N(X) \) are polynomials of \( \{f_j\} \).

Proof: Let \( k \in \mathbb{R}^n \) be in \( F_{\bar{\partial},R} \). This means that there exists a nonzero analytic function \( u(x) \) of Bloch type, i.e. automorphic with a unitary character \( \gamma_k \) of the group \( G \). The absolute value of \( u(x) \) is then \( G \)-periodic and thus can be pushed down to \( M \). Due to compactness of \( M \), it must attain its maximum. Hence, the function \( u(x) \) attains somewhere on \( X \) the maximum of its absolute value. The standard maximum principle now implies that \( u = \text{const} \). This means in particular that \( k = 0 \). Hence, \((1)\) holds.

Representation \((4.4)\) of Lemma 15 implies now that \( A_0(X) \) consists of constants. Indeed, this space is comprised of analytic functions on \( X \) automorphic with respect to unitary characters of \( G \). As we have just shown, every such automorphic function is automatically constant.

Let us extend the \( \bar{\partial} \) operator on \( X \) to the elliptic Dolbeault complex (e.g., [82, Chapter IV, Examples 2.6 and 5.5]):

\[
(6.6) \quad 0 \mapsto \mathcal{A}^{0,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{0,1} \xrightarrow{\bar{\partial}} \mathcal{A}^{0,2} \xrightarrow{\bar{\partial}} ... ,
\]
where $\mathcal{A}^{p,q}$ is the sheaf of germs of smooth $(p,q)$-forms on $X$. The conditions of Theorem 28 are now satisfied, which proves the statement (3).

Finally, the implication (3) $\Rightarrow$ (4) was established in [19].

\[ \square \]

7. LIOUVILLE THEOREMS ON COMBINATORIAL AND QUANTUM GRAPHS

Liouville type theorems have also been studied intensively on discrete objects (graphs, discrete groups) (e.g., [42, 66]). Our technique is insensitive to the local structure of the base of the covering, and hence it can also be applied to abelian co-compact coverings of graphs.

Let $\Gamma$ be a countable non-compact graph with a free co-compact action of an abelian group $G$ of the free rank $n$ and $\tilde{\Gamma}$ be the finite graph $\Gamma/G$. Consider any periodic finite order difference operator $P$ on $\Gamma$. One can define all the basic notions: Floquet solutions, quasimomenta, Fermi variety, etc., exactly the way they were defined before. The same procedure as before leads to the following result that we state without proof:

**Theorem 30.** (1) If the Liouville theorem for the equation $Pu = 0$ holds to an order $N \geq 0$, it holds to any order.

(2) In order for the Liouville theorem to hold, it is necessary and sufficient that the real Fermi surface $F_{P,\mathbb{R}}$ (i.e., the set of unitary characters $\chi$ for which the equation $Pu = 0$ has a non-zero $\chi$-automorphic solution) is finite.

(3) If the Liouville theorem holds, then:

(a) Each solution $u \in V_N(P)$ can be represented as a finite sum of Floquet solutions of order up to $N$.

(b) If $d_N = \text{dim} V_N(P)$, then for all $N \in \mathbb{N}$, we have

$$d_N \leq d_0 q_{n,N} < \infty,$$

where $q_{n,N}$ is the dimension of the space of all polynomials of degree at most $N$ in $n$ variables.

(c) Assume that for each real quasimomentum $k \in F_{P,\mathbb{R}}$ the conditions of Theorem 12 are satisfied. Then for each $N \in \mathbb{N}$ the dimension $d_N$ of the space $V_N(P)$ is equal to

$$\sum_{k \in F_{P,\mathbb{R}}} r_k \left( \binom{n + N}{N} - \binom{n + N - l_0(k)}{N - l_0(k)} \right).$$

Here $r_k$ and $l_0(k)$ are respectively the multiplicity of the zero eigenvalue and the order of the first non-zero Taylor term of the dispersion relation at the point $k \in F_{P,\mathbb{R}}$. The terms
in this sum are the dimensions of the spaces of $\lambda_{\mu(k)}(D)$-harmonic polynomials, and polynomially growing solutions can be described in terms of these polynomials analogously to Theorem 12.

The same can also be done for the so called quantum graphs, where a differential (rather than difference) equation is considered on a graph that is treated as a one-dimensional singular variety rather than a purely combinatorial object (see the details in [51]-[54]). A quantum graph $\Gamma$ is a graph with two additional structures. First of all, the graph must be metric, i.e. each edge $e$ must be supplied with a (finite in our case) length $l_e > 0$ and correspondingly with an ‘arc length’ coordinate. This allows one to do differentiations and integrations and to define differential operators on $\Gamma$. The simplest ones are $-\frac{d^2x}{dx^2} + v(x)$, where differentiation is done with respect to the edge coordinates, and $v(x)$ is a sufficiently nice (e.g., measurable and bounded) potential. In order to define such an operator, one needs to impose some boundary conditions at vertices. All such ‘nice’ conditions have been described (e.g., [38, 48, 54]). The simplest example is the so called Neumann (or Kirchhoff) condition that requires that the functions are continuous at the vertices, and the sum of outgoing derivatives at each vertex is equal to zero. A quantum graph is a metric graph equipped with a self-adjoint operator $P$ of the described kind.

Let now $\Gamma$ be a non-compact quantum graph with a free co-compact isometric action of an abelian group $G$ of the free rank $n$ and $\tilde{\Gamma}$ be the compact quantum graph $\Gamma/G$. We assume that the Hamiltonian $P$ of the quantum graph is $G$-periodic. The basic notions (Floquet solutions, quasimomenta, Fermi variety, etc.) can again be defined for this situation.

**Theorem 31.** The statements of Theorem 30 hold for the quantum graph $G$ under the conditions just described.

8. Further remarks

(1) In most parts of the paper we assumed that all the coefficients of the operators $P$ and $P^*$ are $C^\infty$-smooth. This assumption was made for simplicity only, and in fact one does not need such a strong restriction (e.g., see the discussion in [57] and [50, Section 3.4.D]). For example, for the results of Theorem 27 to hold, it is sufficient (but not truly necessary) that the coefficients of $L$ and $L^*$ are Hölder continuous. It is clear that the conditions on the coefficients could be significantly relaxed, if the operators
were considered in the weak sense, or by means of their quadratic forms. This does not change the general techniques of the proofs, since they rely upon analytic dependence with respect to the quasimomenta, rather than nice properties of coefficients. One only needs to guarantee compactness of the resolvents of the operators $P(\chi)$. However, it was not our intention in this paper to optimize the requirements on the coefficients for all our results to hold.

(2) A stronger statement than Theorem 16 holds: existence of a sub-exponentially growing solution also implies existence of a Bloch one. This is an analog of the Shnol’s theorem (see [27, 36, 77]). It is provided in Theorem 4.3.1 of [50] for the case of periodic equations in $\mathbb{R}^n$, however carrying it over to the case of more general abelian coverings does not present any difficulty. An analogous theorem holds for periodic combinatorial and quantum graphs [55].

(3) The interesting feature of the main results of this text is that they relate the Liouville property to the local behavior of the dispersion relations near the edges of the spectrum of the operator. This threshold behavior is responsible for many things, in particular homogenization [3, 15, 16, 10, 25, 26, 41], structure of the impurity spectrum arising when a periodic medium is locally perturbed [12, 13], Anderson localization, and others. There are less explored issues of this kind, for instance behavior of the Green’s function [8, 56, 68, 84], integral representations of solutions of different classes, and finally Liouville theorems. The last two were also looked upon in our previous paper [57]. The results of the papers quoted above, as well as of this paper, deduce properties of solutions from an assumed spectral edge behavior of the dispersion relation. However, there is very little proven for specific (or even generic) operators about this behavior. The notable exception is the bottom of the spectrum, where much is known (e.g., [44, 74] and recent advances in [15, 17, 78]). An initial study of the internal edges was conducted in the interesting paper [45], however the knowledge in this case remains far from being satisfactory. It is also appropriate to mention that study of analytic properties of the Fermi and Bloch varieties, even for ‘generic’ operators, happens to be a very hard problem (e.g., [34, 35, 46]). There are several unproven conjectures about their generic behavior (e.g., [6, 69]).
As it was mentioned above, the behavior at the bottom of spectrum is responsible for homogenization. Our results on Liouville theorems indicate that in some sense some kind of “homogenization” exists also at interior edges of the spectrum. A recent progress has been made towards precise understanding of what this “homogenization” is by M. Birman for ODEs [11] and M. Birman and T. Suslina for PDEs [16].

An unexpected and counter-intuitive result of our study of Liouville theorems is that they depend on the order $l_0$ of the leading term $\lambda l_0$ of the dispersion relation only, rather than on its full structure. For instance, if $l_0 = 2$, the dimensions are the same as for the Laplace operator, even if $\lambda_2$, the Hessian of the dispersion relation, is degenerate (e.g., the case of the operator $M_t$ with $t = 2\sqrt{3}$ in Section 5). Normally in homogenization theory, the whole term $\lambda l_0$ is important.

Although our results are complete for the case of non-multiple spectral edge ($r = 1$ in terms of Theorem 12 and 18), in the multiplicity case when $r > 1$ we require that the determinant of the leading term $\lambda_0$ of the Taylor expansion is not identically equal to zero. This condition can be modified, by defining the leading term in a different manner, e.g. in the ‘Douglis-Nirenberg sense’ analogous to the corresponding definition of ellipticity [28]. It is not clear, however, whether one can describe the dimensions of the spaces of polynomially growing solutions without any non-degeneracy condition. Fortunately, in practically all cases that are interesting for applications, one expects multiplicity to be absent.

The result of Theorem 29 for the case of Kähler manifold can be extracted from the elliptic case, switching consideration to harmonic functions. Such result was obtained in [19]. This, however, does not work in general and the technique presented in this text is needed.

It is interesting to mention that the dimension (and in fact the geometry) of the underlying manifold $X$ does not explicitly enter our main results. It definitely influences the geometry of dispersion curves and hence the Liouville theorems. However, on the surface it looks like the group $G = \mathbb{Z}^n$ matters more than the manifold. In particular, one can easily cook up a two dimensional covering $X \xrightarrow{Z^n} M$ with an arbitrarily large $n$ (take for example the standard 2-dimensional jungle gym $JG^2$ in $\mathbb{R}^n$, see [73]). Then in the results that concern the dimensions of the
spaces of polynomially growing solutions, one sees mostly the influence of $n$. In particular, one can get large dimensions of these spaces in such a manner. The reason is that, as it has been mentioned before, Liouville theorems are 'homogenized' effects, and being seen from afar, the covering manifold $X$ looks like $\mathbb{R}^n$ [33, 37].

(8) One wonders whether our results could be extended to nilpotent coverings. Some Liouville theorems for harmonic and holomorphic functions do hold in such situation (e.g., [24, 61]). However, our technique so far has not been applicable to the nilpotent case.

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