Identities for \( \sin x \) that came from medical imaging

Peter Kuchment  
Texas A& M University, College Station, TX 77843-3368  
and  
Sergey Lvin  
University of Maine, Orono, ME 04469-5257

Abstract

The article describes some interesting and possibly not previously noticed nonlinear differential identities satisfied by exponential and trigonometric functions. Both formulation and the proofs of these relations do not require mathematics knowledge beyond standard differentiation rules and high school algebra.

Surprisingly, the research that produced these identities was devoted to medical imaging (computerized tomography). A brief description of the origin of these identities is provided.

Relations with some other topics are also briefly mentioned.

1 Introduction

History of sciences is full of examples of unexpected discoveries, when some results appeared as byproducts of an apparently unrelated research. In fact, this can probably be considered as one of the natural features of scientific endeavor. Still, the authors were surprised when their work on medical imaging [11, 12] suddenly produced a series of seemingly new (at least for us) beautiful identities for functions about which “everything is known,” namely standard exponentials and trigonometric, as well as hyperbolic, sines and cosines. It was a little bit depressing when our bold attempt to prove these identities by elementary means, without using the mathematical machinery common
in medical imaging (come on, what can be difficult about $\sin x$, after all!), stalled for a while. We eventually did succeed, albeit as we hope to persuade the reader, there are still things that are not clear about this whole business. Along the way, relations to some interesting theorems from other areas of mathematics have been discovered.

In this short note we would like to present a brief history of the discovery of these identities, as well as how they can be proven by elementary means. Essentially, no knowledge beyond differential Calculus is required to follow the main arguments. In the next section, where we discuss the medical imaging origins of this study, elements of multivariable calculus are used, but these can be skipped without hurting one’s understanding of the main identities and their derivations. The reader who finds this section difficult to read, can skip to the next ones without any harm done.

Some small steps of derivations are formulated as exercises. Some open questions are indicated in the end of the paper.

2 A brief story about medical imaging, Radon transform, and all that

In tomographic medical imaging one sends through the patient’s body some kind of radiation (X-rays, ultrasound waves, electromagnetic waves, etc.), measures the outcome, and tries to use mathematical methods to recover the internal structure of the body. Albeit probably everyone has heard of at least some tomographic procedures (the best-known ones are the X-ray CAT scan and MRI), not everyone realizes the heavy involvement of mathematics in creating the tomographic images. Mathematically speaking, one usually deals with the problem of recovering a function from its integrals over a set of lines (curves) or planes (surfaces). One can think that the function $f(x)$ to be recovered carries some medically important information about the interior of the patient’s body, e.g. $f(x)$ is the density of the tissue at a location $x$. If this function is found, then its density plot would provide a picture that would show to a doctor how a slice of the body looks like. In the simplest cases, the data one measures provides the so-called 2D Radon transform that takes a function $f(x)$ on the plane $^1$ and integrates it over all lines $L$ in the

\footnote{We will not be precise about conditions on functions we deal with here, say they are “very nice”: differentiable as many times as you want and vanishing at large distances.}
plane:

\[ f(x) \mapsto Rf(L) = \int_L f(x)dl. \]

Let us introduce the normal equations of the line \( L \):

\[ x \in L \text{ iff } x \cdot \omega = s, \]

where \( \omega \) is a unit vector normal to \( L \), \( s \) is the (signed) distance from the origin to \( L \), and \( x \cdot \omega \) is the standard dot-product. Thus, the Radon transform \( Rf \) of a function \( f \) can be written as follows:

\[ Rf(\omega, s) = g(\omega, s) := \int_{x \cdot \omega = s} f(x)dl. \]

One can think that \( Rf \) is the data obtained from a scanner, while the unknown “image” is the function \( f \) itself that needs to be recovered from \( Rf \).

One interesting feature of the Radon transform is that its range is rather small in the sense that there are infinitely many conditions a function \( g(\omega, s) \) must satisfy to qualify as the Radon transform of some function \( f \). These are the so-called moment conditions: for any integer \( k \geq 0 \) the \( k \)-th moment of the data

\[ G_k(\omega) := \int_{-\infty}^{\infty} s^k g(\omega, k)ds \]

must be the restriction to the unit circle of vectors \( \omega \) of a homogeneous polynomial of degree \( k \). Indeed, substituting into this formula the definition of the Radon transform \( g = Rf \), one gets

\[ G_k(\omega) := \int_{-\infty}^{\infty} s^k g(\omega, k)ds = \int_{-\infty}^{\infty} s^k \int_{x \cdot \omega = s} f(x)dl ds = \int_{\mathbb{R}^2} (x \cdot \omega)^k f(x)dx. \]

The last expression is clearly a homogeneous polynomial of degree \( k \) of \( \omega \) (since \( x \cdot \omega \) is a homogeneous linear function of \( \omega \)) and thus we have established these so-called “range conditions”. Proving that there are no other range conditions is a different story and takes much more work (see, e.g. [2, 6, 7, 15, 16]).

Why does one want to know these conditions? They are mandatory relations that ideal data collected from a tomographic device must satisfy. Well,
the measured data are never ideal, and so they deviate from these conditions. Thus, knowing the range conditions might be helpful in detecting and correcting some measurement errors. They are of help in other circumstances as well, for instance in completing some missing data (e.g., see [14, 15, 16, 19] and references therein for details).

About seventeen years ago, the authors worked [11, 12] on finding the range conditions for some special Radon-type transform arising in another medical tomography method, so-called SPECT (Single Photon Emission Computed Tomography) [5, 8, 10, 13, 15]. When we found the conditions and proved that we did not miss any, we thought that checking necessity of these conditions (as long as we already knew them) should be very easy, like in the example of the usual Radon transform above. You just plug your transform $Rf$ into the conditions and should be able to see immediately that they are satisfied. Well, when we did this, we saw an infinite series of identities for $\sin x$ (yes, the usual sine!) that we had not seen before and did not know why they should have been true. Although we have found all the necessary proofs and eventually moved to doing other things, the true meaning of these identities has kept us puzzling for all these years.

In this paper we present the formulation and elementary derivation of these identities (not only for $\sin x$, but even for some more “elementary” functions such as linear functions and exponents). We would be happy if someone could shed some more light onto their origin and meaning.

The reader can find more information about mathematical methods of tomography for instance in [4, 5, 6, 8, 9, 10, 13, 15, 16] and in other sources referenced in these publications.

3 Formulation of the identities

We will formulate some differential identities which will involve quite a few derivatives. So, we shall use the shorthand notation $D$ for the derivative $\frac{d}{dx}$. All elementary functions that we are going to consider, satisfy simple differential equations

$$ Du = \lambda u \tag{1} $$

or

$$ D^2 u = \lambda^2 u, \tag{2} $$

where $\lambda$ is a (possibly complex) number. For instance, solutions of equation (1) for $\lambda = 0$ are constants, while for non zero values of $\lambda$ they give us expo-
nential functions. Analogously, solutions of (2) for $\lambda = 0$ are linear functions, while for non zero values of $\lambda$ they in particular give us trigonometric and hyperbolic sine and cosine.

We now formulate the main results.

**Theorem 1.** Any solution $u$ of the first-order equation $Du = \lambda u$ satisfies for any natural $n$ the following identity:

$$\sum_{k=0}^{n} \binom{n}{k} (D - u) \circ (D - u + \lambda) \circ \ldots \circ (D - u + (k - 1)\lambda) u^{n-k} = 0. \quad (3)$$

**Theorem 2.** Any solution $u$ of the second-order equation $D^2 u = \lambda^2 u$ satisfies for any odd natural $n$ the identity (3).

Before going any further, let us explain the meaning of the expressions involved into (3). First of all, $\binom{n}{k}$ is the standard notation for the binomial coefficient “$n$ choose $k$.” The expression $u^{n-k}$ at the very right is just the function $u$ raised to the power $n - k$. Now, expressions like $(D - u)$ are considered as operations (operators), i.e. they need to be applied to a function. For instance, $(D - u)$ applied to a function $f$ produces $(D - u)f = \frac{df}{dx} - u(x)f(x)$. The little circles $\circ$ mean composition of these operators from right to left. For instance,

$$(D - u) \circ (D - u + \lambda)f = (D - u)(\frac{df}{dx} - u(x)f(x) + \lambda f(x))$$

$$= \frac{d^2 f}{dx^2} - \frac{d(u f)}{dx} + \lambda \frac{df}{dx} - u(x) \frac{df}{dx} + u^2(x)f(x) - \lambda u(x)f(x).$$

So, the expression in (3) directs us to raise $u$ to the power $(n-k)$, then apply to it the operator $(D - u + (k-1)\lambda)$, then to apply to the resulting function the operator $(D - u + (k-2)\lambda)$, etc. To avoid confusion in the case of $k = 0$, it might be easier to think that the composition in (3) has $k$ operator factors. In other words, the term for $k = 0$ does not have any of these factors and thus is just $u^n$.

An important thing to notice is that the operations $D$ and $u$ (differentiation and multiplication by $u$) do not commute, i.e. the result of their composition depends on its order.
Exercise 3. Check that \((D \circ u)f = \frac{d(uf)}{dx}\) and \((u \circ D)f = u(x)\frac{df}{dx}\) produce different results.

If this were not so, the identities (3) would become trivial, as we will see a little bit later.

A side comment is that the non-commutativity of operations of differentiation with respect to \(x\) and multiplication by \(x\) happens to play a crucial role in quantum mechanics and in a sense is responsible for the weird properties and magnificent applications of this wonderful theory.

Since solutions of the equations (1)-(2) are very simple functions (constants and exponential, linear, trigonometric or hyperbolic functions), let us look at some examples to see whether the identities (3) might be trivial, or at least look familiar.

Take (1) for \(\lambda = 0\), so the solution \(u\) is just a constant \(C\). Then (3) looks as follows:

\[
\sum_{k=0}^{n} \binom{n}{k} (D - C)^k C^{n-k} = 0.
\]

Exercise 4. (This is the particular case of Theorem 1 when \(\lambda = 0\).) Prove this identity for any constant \(C\) and natural \(n\). (Hint: use the fact that derivatives of constants are zeros and then some high school algebra.)

Consider now (1) for \(\lambda = 1\) and \(u = e^x\). Then (3) reduces to

\[
\sum_{k=0}^{n} \binom{n}{k} (D - e^x) \circ (D - e^x + 1) \circ \ldots \circ (D - e^x + (k-1)) e^{(n-k)x} = 0.
\]

Hmmm... Does not look familiar? Well, we will not prove this, since we are going to prove the more difficult Theorem 2. But,

Exercise 5. After reading this paper, provide the proofs of the Theorem 1 when \(\lambda \neq 0\) (this is easier than the proof of Theorem 2 you’ll read). You have already considered the case \(\lambda = 0\) in the previous exercise.

Let us look at an example of a solution of the second-order equation \(D^2u = \lambda^2 u\), say take \(u = \sin x\) and \(\lambda = i\). Then the identity (3) for \(n = 3\) becomes (using more familiar \(\frac{d}{dx}\) instead of \(D\))

\[
\sin^3 x + 3(\frac{d}{dx} - \sin x) \sin^2 x + 3(\frac{d}{dx} - \sin x) \circ (\frac{d}{dx} - \sin x + i) \sin x \\
+ (\frac{d}{dx} - \sin x) \circ (\frac{d}{dx} - \sin x + i) \circ (\frac{d}{dx} - \sin x + 2i) 1 = 0.
\]
Obvious, isn’t it? Well, do not despair if you do not see it right away, we don’t either.

As you will see, the direct proofs of Theorem 2 (at least the proofs that we know) are rather different when \( \lambda = 0 \) and \( \lambda \neq 0 \). Certainly, it is not hard to derive the identities for \( \lambda = 0 \) from the ones for \( \lambda \neq 0 \) by taking the limit when \( \lambda \to 0 \).

**Exercise 6.** Justify (3) for \( u = x \) and \( \lambda = 0 \) by taking the limit when \( \lambda \to 0 \) in (3) for \( \lambda \neq 0 \).

We, however, would like to see direct algebraic proofs of the identities for both cases \( \lambda = 0 \) and \( \lambda \neq 0 \). Such proofs are provided below.

### 4 Proof of Theorem 2 for \( \lambda = 0 \)

We assume now that \( \lambda = 0 \) and thus the function \( u \) satisfies the equation \( D^2u = 0 \). In this case the identities (3) that we intend to prove simplify to

\[
\sum_{k=0}^{n} \binom{n}{k} (D-u)^k u^{n-k} = 0
\]

for any odd natural \( n \).

**Exercise 7.** Check that this identity fails for \( n = 2 \).

One can understand (5) also as follows (persuade yourself that this is the case):

\[
\left( \sum_{k=0}^{n} \binom{n}{k} (D-u)^k \circ u^{n-k} \right) 1 = 0.
\]

Does the expression in the parentheses remind you of anything? Yes, you are right, it resembles the familiar binomial expression

\[
\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k},
\]

where \( a = D - u \) and \( b = u \). Then, according to the binomial formula, the whole thing probably should be equal to \( (a+b)^n = (D-u+u)^n 1 = D^n 1 = \frac{d^n 1}{dx^n} = 0 \) (since any derivative of 1 is zero). Done! Well, not so fast. First
of all, since $D$ and $u$ do not commute, the usual binomial formula does not apply. Besides, it looks like we obtained the identity (3) for all natural $n$, which is not correct, as we saw in Exercise 7. So, let us move slower.

If $\lambda = 0$, then function $u$ is linear. Let us concentrate on the case when $u(x) = x$.

**Exercise 8.** Generalize to arbitrary linear functions $u = ax + b$ our proof given below for the case $u = x$. (Alternatively, you might try to derive the result for any linear function from the one for $u = x$.)

Let us first of all rewrite our identity (6) for the case when $u = x$:

$$
\sum_{k=0}^{n} \binom{n}{k} (D - x)^k x^{n-k} = 0.
$$

(7)

We will now introduce a nice general trick that will enable us to rewrite (7) in a slightly different form. It presents a simple instance of the so-called **gauge transformations** arising in many problems of mathematical physics, e.g. when studying Schrödinger operators of quantum mechanics in presence of magnetic fields, Maxwell equations of electromagnetics, etc. It is based on the following simple identity:

$$
(D - x)^k \left( e^{\frac{x^2}{2}} f(x) \right) = e^{\frac{x^2}{2}} D^k f(x).
$$

(8)

Do you see what it does? Commutation with the function $e^{\frac{x^2}{2}}$ kills the term $-x$ added to the derivative and $(D - x)^k$ becomes $D^k$:

$$
e^{-\frac{x^2}{2}} \circ (D - x)^k \circ e^{\frac{x^2}{2}} = D^k.
$$

This immediately implies the following

**Lemma 9.** The identities (7) are equivalent to

$$
\left( \sum_{k=0}^{n} \binom{n}{k} D^k \circ x^{n-k} \right) e^{-\frac{x^2}{2}} = 0.
$$

(9)

**Exercise 10.** 1. Prove the identity

$$
(D + x) e^{-\frac{x^2}{2}} = 0.
$$

(10)
2. Prove (8).


So, now the proof of Theorem 2 for $\lambda = 0$ boils down to proving (9) for odd $n$. Notice that if $D$ and $x$ commuted, then the expression in parentheses in (9) would become just $(D + x)^n$, and thus, due to (10), we would immediately obtain the validity of (9). However, life is not so easy and thus the operators do not commute (besides, as we already know, (9) holds for odd $n$ only).

Based on (9), let us introduce the differential operators

$$P_n(D, x) = \sum_{k=0}^{n} \binom{n}{k} D^k \circ x^{n-k}.$$  

The punch line is in the following statement:

**Lemma 11.** The following recurrence relation holds:

$$P_{n+2}(D, x) = P_{n+1}(D, x) \circ (D + x) + (n + 1)P_n(D, x).$$  (11)

**Proof of the Lemma.** We will compute $P_{n+1}(D, x) \circ (D + x)$ and show that it coincides with $P_{n+2}(D, x) - (n + 1)P_n(D, x)$. In order to do so, we will use the easy to verify commutation relation

$$D \circ x^m = x^m \circ D + mx^{m-1}$$

and its consequence

$$x^m \circ D = D \circ x^m - mx^{m-1}.$$  (12)

**Exercise 12.** Prove these relations.

Now, take a deep breath, and ...

$$P_{n+1}(D, x) \circ (D + x) = \left( \sum_{k=0}^{n+1} \binom{n+1}{k} D^k \circ x^{n+1-k} \right) \circ (D + x)$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} D^k \circ x^{n+1-k} \circ D + \sum_{k=0}^{n+1} \binom{n+1}{k} D^k \circ x^{n+2-k}.$$  (13)

Let us now use (12) to commute $x^{n+1-k}$ with $D$ in the first sum to get

$$\sum_{k=0}^{n+1} \binom{n+1}{k} D^{k+1} \circ x^{n+1-k} + \sum_{k=0}^{n+1} \binom{n+1}{k} D^k \circ x^{n+2-k}$$

$$- \sum_{k=0}^{n+1} (n + 1 - k) \binom{n+1}{k} D^k \circ x^{n-k}.$$  (14)
The first two sums can be rewritten as

\[
\sum_{k=1}^{n+2} \binom{n+2}{k-1} D^{k+1} \circ x^{(n+2)-k} + \sum_{k=0}^{(n+2)-1} \binom{n+2}{k} D^k \circ x^{(n+2)-k} = \sum_{k=0}^{n+2} \binom{n+2}{k} D^k \circ x^{(n+2)-k} = P_{n+2}(D, x).
\]

We used here the standard property of the Pascal triangle:

\[
\binom{n+2}{k-1} + \binom{n+2}{k} = \binom{n+2}{k}.
\]

It only remains to handle the remainder

\[
\sum_{k=0}^{n+1} (n + 1 - k) \binom{n+1}{k} D^k \circ x^{n-k}.
\]

Exercise 13. Prove that this expression is equal to \((n + 1)P_n(D, x)\). (Hint: this only requires the knowledge of what binomial coefficients are.)

This finishes the proof of the Lemma.

Corollary 14. For any odd natural \(n\), the following factorization holds with some operator \(Q(D, x)\):

\[
P_n = Q \circ (D + x).
\]

Proof. Let us prove this by induction. When \(n = 1\), we have \(P_1 = D + x\), and the statement is obvious. Assume that we have proven it for some odd \(n\), i.e. \(P_n = Q \circ (D + x)\). Then the previous Lemma implies that \(P_{n+2} = (P_{n+1} + (n + 1)Q) \circ (D + x)\).

Exercise 15. Show that the factorization \(P_n = Q \circ (D + x)\) fails for even \(n\) (check \(n = 2\)).

Now the proof of Theorem 2 for \(\lambda = 0\) is immediate. Indeed, for any odd \(n\), the left hand side in the identity (9) in question becomes \(P_n e^{-\frac{x^2}{2}} = Q \circ (D + x)e^{-\frac{x^2}{2}}\). Applying (10), we conclude that this is zero. This finishes the proof of the Theorem.
5 Proof of Theorem 2 for $\lambda \neq 0$

Let us now outline the steps of the proof of Theorem 2 in case $\lambda \neq 0$. More details can be found in [11].

**Step 1.** (In fact, here we prove Theorem 1.) Suppose first that $u$ satisfies the first-order equation $Du = \lambda u$ (and thus certainly $D^2 u = \lambda^2 u$ as well). We can use another gauge transform to simplify our identities. Instead of commuting with the Gaussian exponential function as before, we will commute with a power of $u$. Namely, we will use the identity

$$u^{-m}(D - u)u^m = D - u + m\lambda.$$  \hspace{1cm} (17)

**Exercise 16.** Prove (17).

We can now use (17) in each factor of (3) to rewrite it as follows:

$$\left(\sum_{k=0}^{n} \binom{n}{k} [(D - u) \circ u^{-1}]^k\right)u^n = 0.$$  \hspace{1cm} (18)

Here we notice that we deal with an operator binomial $\sum_{k=0}^{n} \binom{n}{k} A^k$, where $A = (D - u) \circ u^{-1}$. Since only one operator is involved, no non-commutativity arises, and thus the usual binomial formula works. This reduces identity (3) to

$$[(D - u) \circ u^{-1} + 1]u^n = (D \circ u^{-1})u^n = 0.$$  \hspace{1cm} (19)

Now an immediate calculation shows that $(D \circ u^{-1})u^n = 0$ holds.

Careful readers might object to our calculations in this section that involve negative powers of $u$, as well as positive ones. However, this algebraic problem can be overcome [11].

**Step 2.** Suppose $Du = -\lambda u$. Then (3) is true for odd $n$, because one can easily check that “symmetric” terms corresponding to $k$ and $n - k$ cancel.

**Step 3.** Let us now suppose that $D^2 u = \lambda^2 u$. If our identities were linear with respect to $u$ (which they are not), then the cases when $Du = \pm \lambda u$ (discussed in the two previous steps) would suffice, since any solution could be expanded into a sum of these two. Nonlinearity seems to destroy this idea. However, having nothing better in mind, let us still try. Thus, $u = v + w$, where $Dv = -\lambda v$ and $Dw = \lambda w$ (the assumption $\lambda \neq 0$ is critical for this decomposition$^2$).

$^2$This is just standard exponential representation of solutions studied in ODEs (or, if you will, the representation of $\sin x$ as combination of $e^{\pm ix}$).
Step 4. Let us use the same commutation trick (17) as before, but using
commutation with $w$ rather than $u$. Then the left-hand side of (3) can be
written as
\[
\sum_{k=0}^{n} \binom{n}{k} (D - v - w) \circ ... \circ (D - v - w + (k - 1)\lambda)(v + w)^{n-k} \\
= \sum_{k=0}^{n} \binom{n}{k} [(D - v) \circ w^{-1} - 1]^k(vw^{-1} + 1)^{n-k} \cdot w^n.
\]

(20)

Step 5. Let us introduce the following operator notations: $A = (D - v) \circ w^{-1}$ and $B = vw^{-1}$. Then the last sum becomes
\[
\sum \binom{n}{k} (A - 1)^k(B + 1)^{n-k}.
\]

If the operators $A$ and $B$ commuted, then according to the binomial formula
this would boil down to $(A + B)^n$ and thus also to $\sum \binom{n}{k} A^k B^{n-k}$. The
interesting thing is that the latter conclusion holds even without commuta-
tivity:

Lemma 17. For any two operators $A, B$ the following equality holds:
\[
\sum \binom{n}{k} (A - 1)^k(B + 1)^{n-k} = \sum \binom{n}{k} A^k B^{n-k}.
\]

Exercise 18. Prove this lemma. (Hint: this equality does not require chang-
ing the order of the factors, and thus one can show that if it holds for com-
muting operators, then also for non-commuting ones.)

How now this can help us with the proof of the theorem? It allows us to
drop the terms $\mp 1$ in (20) to get
\[
\sum_{k=0}^{n} \binom{n}{k} [(D - v) \circ w^{-1}]^k(vw^{-1} + 1)^{n-k} \cdot w^n.
\]

Now one can reverse Step 4 (undoing the commutations with powers of $w$ we
have done) and rewrite (20) as
\[
\sum_{k=0}^{n} \binom{n}{k} (D - v) \circ ... \circ (D - v + (k - 1)\lambda)v^{n-k},
\]
which is equal to zero due to the Step 2, since $v$ solves the first-order equation
$Dv = -\lambda v$. This proves Theorem 2. \qed
6 Final remarks and acknowledgments

• For the readers with a good knowledge of algebra and a taste for generalizations, we can mention that the identities (3) hold in a much more general situation. Namely, \( u \) can be assumed to be an element of a commutative algebra with cancelation and with differentiation \( D \) over a field \( \Lambda \). Then \( \lambda \) should be an element of the field \( \Lambda \) [11].

• Generalizations of some of the identities we discussed are available:

Theorem 19. If \( u \) satisfies \( D^2 u = 0 \), then the identity

\[
\sum_{k=0}^{n} \binom{n}{k} (D - u)^k \circ D^m u^{n-k} = 0
\]

holds for all odd integer \( n \geq 1 \) and even \( m \geq 0 \).

• It is surprising that the identities (3) have been related to the so-called separate analyticity theorems (Hartogs-Bernstein theorems) in several complex variables [1]. For instance, the following amazing theorem is essentially equivalent to these identities [1, 17, 18]:

Theorem 20. Let \( f(x) \) be a continuous function in the exterior of a disk in the real plane \( \mathbb{R}^2 \) and such that its restriction to each tangent line to the disk extends to an entire function of one complex variable. Then \( f \) extends to an entire function in \( \mathbb{C}^2 \).

More discussion of the relations of the range conditions with complex analysis can be found for instance in [3, 10, 11, 12].

• It would be interesting to find an algebraic proof of the identities that would work simultaneously for \( \lambda = 0 \) as well as for \( \lambda \neq 0 \).

• It is clear that the identities discussed must be related to special function theory and group representations. It would be interesting to find such relations.

• The formulations of the Theorems 1 and 2 lead to the natural question: what can be said for solutions of the equation of 3rd order \( D^3 u = \lambda^3 u \) and higher? A natural guess would be that the same identities hold,
but only for an arithmetic sequence of numbers $n$ with the difference equal to 3. It is hard to compute these expressions by hand even for small values of $n$, say for $n = 4$. Ms. E. Rodriguez, a former Masters student of the first author, has used the Maple symbolic algebra system to check this conjecture. The result is negative, the natural conjecture fails for equations of third order [20]. So, what (if anything) happens to solutions of higher order differential equations $D^m u = \lambda^m u$? Are they deprived of any such identities? We do not know the answer to this question.

The authors express their gratitude to S. Fulling, D. Hensley, Z. Sunik and G. Tee for useful comments.

References


