1. (10) Define the following terms

(a) Suppose \( f(x) \) is continuous at all points on the interval \([0, 6]\), except at \( x = 2 \).

i. How is \( \int_0^2 f(x) \, dx \) defined?

\[
\int_0^2 f(x) \, dx = \lim_{\epsilon \to 0^+} \int_0^{2-\epsilon} f(x) \, dx
\]

The integral is said to converge if and only if the limiting value is finite.

ii. How is \( \int_0^4 f(x) \, dx \) defined?

\[
\int_0^4 f(x) \, dx = \lim_{\epsilon \to 0^+} \int_0^{2-\epsilon} f(x) \, dx + \lim_{\epsilon \to 0^+} \int_{2+\epsilon}^4 f(x) \, dx
\]

The integral is said to converge if and only if both limits exist and are finite.

(b) The trapezoid rule for approximating \( \int_a^b f(x) \, dx \) is given by

\[
\int_a^b f(x) \, dx \approx \frac{h}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]
\]

Explain why this is called the trapezoid rule, and how the above expression is obtained. Be sure to tell me what \( h \) is.

The interval of integration \([a, b]\) is divided into \( n \) equal size subintervals each of length \( h = (b-a)/n \). The "area" under the graph of \( f \) between the partition points \( x_{i-1} \) and \( x_i \) is approximated with the area of the trapezoid whose height is \( h \) and whose base lengths are \( f(x_{i-1}) \) and \( f(x_i) \). The area of this trapezoid is \( \frac{h}{2} \left( f(x_{i-1}) + f(x_i) \right) \).

The areas of these individual trapezoids are then added together and we arrive at the approximation

\[
\int_a^b f(x) \, dx \approx h \left( \frac{f(x_0) + f(x_1)}{2} + h \frac{f(x_1) + f(x_2)}{2} + \cdots + h \frac{f(x_{n-1}) + f(x_n)}{2} \right)
\]

\[
= \frac{h}{2} \left[ f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + f(x_n) \right]
\]

The fact that this sum converges to the value of the definite integral follows from the fact that it is the average of the left and right Riemann sums for the given partition.
2. (10) State the mean value theorem for a function \( f(x) \), and then use it to prove that if \( f(1) = 0 \) and \( f'(x) < 0 \) in the interval \((1/2, 3/2)\) then \( f(x) > 0 \) if \( 1/2 < x < 1 \).

The statement of the mean value theorem is the following: Let \( f(x) \) be continuous on the interval \([a, b]\) and differentiable on the interval \((a, b)\). Then there is a point \( \xi \) in the open interval \((a, b)\), that is \( a < \xi < b \), such that

\[
f(b) - f(a) = f'(\xi)(b - a).
\]

To verify that \( f(x) > 0 \) for any \( x \) between 1/2 and 1, we write

\[
f(x) = f(x) - f(1) = f'(\xi)(x - 1).
\]

Since \( x \) is between 1/2 and 1 and the point \( \xi \) lies between \( x \) and 1, we know that \( \xi \) is also between 1/2 and 1. Thus, \( f'(\xi) < 0 \), and \( x - 1 < 0 \). Hence \( f(x) \) their product must be positive.

3. (30) Evaluate the following integrals:

(a) \( \int t \ln t \, dt \)

Evaluate this using integration by parts. Set \( u = \ln t \) and \( dv = t \, dt \). Then

\[
\int t \ln t \, dt = \frac{t^2}{2} \ln t - \int \frac{t^2}{2} \, dt = \frac{t^2}{2} \ln t - \frac{t^2}{4} + c.
\]

(b) \( \int \tan^2 x \, dx \)

\[
\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + c
\]

(c) \( \int \frac{x^2 + x}{x^2 - x} \, dx \)

\[
\int \frac{x^2 + x}{x^2 - x} \, dx = \int \frac{x + 1}{x - 1} \, dx = \int \left(1 + \frac{2}{x - 1}\right) \, dx = x + 2 \ln |x - 1| + c
\]
(d) $\int_{1}^{4} e^{\sqrt{x}} \, dx$

Make the substitution $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}} \, dx$, or $2u \, du = dx$ first, and then integrate by parts.

$$\int_{1}^{4} e^{\sqrt{x}} \, dx = \int_{1}^{2} 2ue^u \, du = 2ue^u|_{1}^{2} - \int_{1}^{2} 2e^u \, du$$

$$= (4e^2 - 2e) - (2e^2 - 2e)$$

$$= 2e^2$$

(e) $\int_{0}^{3} \frac{dx}{\sqrt{9 + x^2}}$

Make the substitution $x = 3 \tan \theta$, $dx = 3 \sec^2 \theta \, d\theta$.

$$\int_{0}^{3} \frac{dx}{\sqrt{9 + x^2}} = \int_{0}^{\pi/4} \frac{3 \sec^2 \theta}{\sqrt{9 + 9 \tan^2 \theta}} \, d\theta = \int_{0}^{\pi/4} \frac{3 \sec^2 \theta}{\sqrt{1 + \tan^2 \theta}} \, d\theta$$

$$= \int_{0}^{\pi/4} \sec \theta \, d\theta = (\ln |\sec \theta + \tan \theta|)|_{0}^{\pi/4}$$

$$= \ln \left| \sqrt{2} + 1 \right| - \ln |1|$$

$$= \ln \left( \sqrt{2} + 1 \right)$$

4. (15) Find the area bounded by the curves $y = \sin x$ and $y = \sin^2 x$ between the lines $x = 0$ and $x = \pi$.

The first question is which function is larger than the other. Since for $x$ in the interval $[0, \pi]$, $0 \leq \sin x \leq 1$, we have $\sin^2 x \leq \sin x$. Thus, the areal equals

$$\text{area} = \int_{0}^{\pi} (\sin x - \sin^2 x) \, dx$$

$$= \int_{0}^{\pi} \sin x \, dx - \int_{0}^{\pi} \frac{1 - \cos 2x}{2} \, dx$$

$$= \left[ -\cos x - \frac{x}{2} + \frac{\sin 2x}{4} \right]_{0}^{\pi}$$

$$= \left( 1 - \frac{\pi}{2} \right) - (-1) = 2 - \frac{\pi}{2}$$
5. (10) What is the length of the following curve 

\[ x(t) = t^2 - 1, \quad y(t) = \frac{1}{1 + t}, \quad 0 \leq t \leq 1 \]

You do not have to evaluate the integral that results from your setup.

\[
\text{length} = \int_{0}^{1} \left[ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 \right]^{1/2} dt
\]

\[
= \int_{0}^{1} \left[ (2t)^2 + \left( \frac{-1}{(1 + t)^2} \right)^2 \right]^{1/2} dt
\]

\[
= \int_{0}^{1} \left[ 4t^2 + \frac{1}{(1 + t)^4} \right]^{1/2} dt
\]

6. (10) Solve the following initial value problem

\[ \frac{dy}{dx} = (x - 1) y^2, \quad y(0) = 2. \]

\[
\frac{dy}{dx} = (x - 1) y^2
\]

\[
\frac{1}{y^2} \frac{dy}{dx} = x - 1
\]

\[
\int \frac{1}{y^2} \frac{dy}{dx} \, dx = \int (x - 1) \, dx
\]

\[
-\frac{1}{y} = \frac{x^2}{2} - x + c
\]

To determine \( c \) set \( t = 0 \) and \( y = 2 \). This gives the equation

\[ -\frac{1}{2} = c \]

Thus,

\[
y = -\frac{1}{\frac{x^2}{2} - x - \frac{1}{2}}
\]

\[
= \frac{2}{1 + 2x - x^2}
\]
7. (15) Simpsons rule (remember that \( n \) must be even) is

\[
\int_a^b f(x) \, dx \approx \frac{h}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)].
\]

The error in approximating the integral with the expression above is bounded by

\[
|\text{error}| \leq \frac{K (b-a)}{180} h^4,
\]

where the constant \( K \) is such that \( |f^{(4)}(x)| \leq K \) for all \( x \) in the interval of integration.

(a) With \( n = 4 \) use Simpson’s rule to find an approximate value of

\[
\int_0^2 \frac{\sin \pi x}{1+x} \, dx
\]

Make a table of data first

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<th>1</th>
<th>2</th>
<th>3</th>
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<td>0</td>
<td>-2/5</td>
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</tr>
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</table>

Thus,

\[
\int_0^2 \frac{\sin \pi x}{1+x} \, dx \approx \frac{h}{3} [y_0 + 4y_1 + 2y_2 + 4y_3 + y_4]
\]

\[
= \frac{1/2}{3} [0 + 4(2/3) + 2(0) + 4(-2/5) + 0]
\]

\[
= \frac{1}{6} \left[ \frac{16}{15} \right] = \frac{8}{45} \approx 0.178
\]

(b) It can be shown that \( K = 55 \) satisfies the condition \( |f^{(4)}(x)| \leq K \) on the interval \([0,2]\).

Estimate the error.

\[
|\text{error}| \leq \frac{K (b-a)}{180} h^4 = \frac{55 (2)}{180} \left( \frac{1}{2} \right)^4
\]

\[
= \frac{11}{288} \approx 0.038
\]

This tells us that the value of the integral lies in the interval

\[
0.14 = 0.178 - 0.038 < \int_0^2 \frac{\sin \pi x}{1+x} \, dx < 0.178 + 0.038 = 0.216
\]
(c) Using $K = 55$, determine how large $n$ must be in order to ensure that the error will be less than $10^{-6}$.

We need to pick $n$ so that

$$|\text{error}| \leq \frac{K(b-a)}{180}h^4 = \frac{55}{180} \left(\frac{2}{n}\right)^4 \leq 10^{-6}$$

or

$$\frac{55}{180} \cdot 2^410^6 \leq n^4$$

or

$$n \geq \left(\frac{55}{180} \cdot 2^410^6\right)^{1/4} = 2 \left(\frac{11}{18}\right)^{1/4} 10^{3/2} \geq 55.9$$

So $n \geq 56$ will give us an error less than $10^{-6}$.