1. (20) Define the following terms:

(a) Statement (in the context of logic)
   A statement is a declarative sentence, which is either true or false.

(b) The Cartesian product of two sets \( A \) and \( B \).
   The Cartesian product of \( A \) and \( B \) is
   \[ A \times B = \{ (a, b) : a \in A \text{ and } b \in B \} \]

(c) For each \( \alpha \in \Lambda \), let \( A_\alpha \) be a set. Define \( \cap_{\alpha \in \Lambda} A_\alpha \).
   This set is defined as
   \[ \cap_{\alpha \in \Lambda} A_\alpha = \{ x : x \in A_\alpha \text{ for each } \alpha \in \Lambda \} \]

(d) If \( P \) and \( Q \) are statements, define the statement \( P \implies Q \). Note: a truth table is much preferred.

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2. (15) State Peano’s axioms, and give an example of a set that does not satisfy these axioms.

The axioms are posted on the web page for this course. An example of a set that does not satisfy the axioms is \( S = \{ a \} \). Since there is not a second element in this set \( a + 1 \) is not defined, so \( S \) does not satisfy the axioms.

3. (20) For each of the statements below decide if they are true or false. If a statement is true prove it, and if it’s false supply a counter example.

(a) \( A \cup B = A \cap B \).
   This statement is false. An example to demonstrate this is: set the universal set equal to \( N \) the natural numbers. Set \( A = \{ 1 \} \) and \( B = \{ 2 \} \). Then \( \overline{A \cup B} = \{ 3, 4, \cdots \} \), and \( A \cap B = \emptyset \).

(b) \( P \implies Q \) is logically equivalent to \( P \land (\neg Q) \).
   This statement is false. If you look at the truth tables for these two statements this becomes clear

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4. (15) Use induction to verify the formula

\[ \sum_{i=1}^{n} (2i - 1) = n^2. \]

Set \( P = \left\{ n \in N : \sum_{i=1}^{n} (2i - 1)^2 = n^2 \right\} \). To see that \( 1 \in P \), we evaluate both sides of the formula with \( n = 1 \). The RHS equals 1, and the LHS equals

\[ \sum_{i=1}^{1} (2i - 1) = (2 - 1) = 1. \]

Thus, \( 1 \in P \). Assume that \( n \in P \). Then we have

\[
\sum_{i=1}^{n+1} (2i - 1) = \sum_{i=1}^{n} (2i - 1) + (2n + 1) = n^2 + 2n + 1 = (n+1)^2.
\]

Thus, \( n + 1 \in P \), and the induction axiom tells us that \( P = N \).

5. (10) Let \( A \) be a set, and let \( P(A) \) denote the power set of \( A \). Let \( |A| \) denote the number of elements in the set \( A \).

(a) If \( A = \{1, q, 5\} \), what is \( P(A) \).

\[ P(A) = \{\emptyset, \{1\}, \{q\}, \{5\}, \{1, q\}, \{1, 5\}, \{5, q\}, \{1, q, 5\}\}. \]

(b) Prove the following formula

\[ |P(A)| = 2^{|A|}. \]

One way to prove this is by induction. So let

\[ P = \left\{ n \in N : |P(A)| = 2^{|A|}, \text{ where } |A| = n \right\}. \]

It is easy to see that \( 1 \in P \). So assume that \( n \in P \). Let \( A \) be a set with \( n + 1 \) elements. Denote one of these elements by \( a \). Any subset of \( A \) either contains the element \( a \) or it doesn’t. The number of subsets that do not contain \( a \) are the same as the number of subsets of a set with \( n \) elements. By the assumption that \( n \in P \), there are \( 2^n \) such subsets. Any subset of \( A \) that contains \( a \) can be obtained by adding \( a \) to a subset of \( A \) that does not contain \( a \); there are \( 2^n \) such subsets. Thus, the number of subsets of \( A \) equals

\[ |P(A)| = 2^n + 2^n = 2^{n+1}, \]

and we see that \( n + 1 \in P \). By induction axiom we have \( P = N \).
6. (20) A function \( f(x) \) is said to be ambivalent with respect to \( l \) at the point \( x = a \) if

\[
\forall \epsilon > 0, \forall \delta > 0, \exists x_1 \text{ and } \exists x_2 \text{ such that } \\
|x_1 - a| < \delta, \; |x_2 - a| < \delta, \; |f(x_1) - l| < \epsilon \text{ and } |f(x_2) - l| > \epsilon
\]

(a) What does it mean to say that \( f \) is not ambivalent with respect to \( l \) at the point \( a \).

\[
\exists \epsilon > 0, \exists \delta > 0, \forall x_1 \text{ and } \forall x_2 \\
|x_1 - a| \geq \delta, \text{ or } |x_2 - a| \geq \delta, \text{ or } |f(x_1) - l| \geq \epsilon, \text{ or } |f(x_2) - l| \leq \epsilon
\]

(b) Find, if possible, an example of a function \( f \) that is ambivalent with respect to \( 2 \) at the point \( x = 1 \).

The condition \( |f(x_1) - l| \geq \epsilon \) implies that \( f \) must take on values arbitrarily far from \( l \), as well as taking on value arbitrarily close to \( l \). So let's try

\[
f(x) = \begin{cases} \\
2, & x = 1 \\
\frac{1}{x - 1}, & x \neq 1
\end{cases}
\]

Then for any \( \epsilon > 0 \) set \( x_1 = 1 \). Then we have

\[
|x_1 - 1| = 0 < \delta \text{ and } |f(x_1) - l| = |2 - 2| = 0 < \epsilon
\]

To see that we can find a value \( x_2 \) that satisfies the other two conditions, note that the expression \( \frac{1}{x - 1} \) can be made arbitrarily large by picking \( x \) close to 1. So for any \( \epsilon \) and \( \delta \), which are positive pick \( x_2 \) so that

\[
0 < x_2 - 1 < \min \{ \delta, 1/\epsilon, 1 \}.
\]

For such a number we have

\[
|x_2 - a| = |x_2 - 1| = x_2 - 1 < \delta \text{ and } \\
|f(x_2) - l| = \left| \frac{1}{x_2 - 1} - 2 \right| \\
= \left| \frac{3 - 2x_2}{x_2 - 1} \right| > \frac{|3 - 2x_2|}{1/\epsilon} \\
= \epsilon |3 - 2x_2| > \epsilon
\]

Note, since \( 0 < x_2 - 1 < 1 \), we have \( |3 - 2x_2| > 1 \).