1. (15) Define the following

(a) \( f(x, y) \) is differentiable at the point \((1, -2)\).

This means that both of \( f \)'s partial derivatives exist at the point \((1, -2)\), and for \( \Delta x \) and \( \Delta y \) small enough we have

\[
f(1 + \Delta x, -2 + \Delta y) = f(1, -2) + \left. \frac{\partial f}{\partial x} \right|_{(1, -2)} \Delta x + \left. \frac{\partial f}{\partial y} \right|_{(1, -2)} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,
\]

where

\[
\lim_{(\Delta x, \Delta y) \to (0,0)} \epsilon_i = 0, \text{ for } i = 1 \text{ or } 2.
\]

(b) \( \left. \frac{\partial f}{\partial x} \right|_{(2, 3)} \)

\[
\left. \frac{\partial f}{\partial x} \right|_{(2, 3)} = \lim_{h \to 0} \frac{f(2 + h, 3) - f(2, 3)}{h}
\]

(c) What do the spherical variables \( \rho, \theta, \) and \( \phi \) represent?

If \((x, y, z)\) are the Cartesian coordinates of a point \( P \), then \( \rho = \sqrt{x^2 + y^2 + z^2} \), the distance of \( P \) to the origin. The variable \( \phi, 0 \leq \phi \leq \pi \) is the smaller of the two angles the line joining the origin to the point \( P \) makes with the positive \( z \)-axis, and the variable \( \theta, 0 \leq \theta \leq 2\pi \) is the angle the line in the \( x-y \) plane from the origin to the point \((x, y, 0)\) makes with the positive \( x \)-axis, with a counter clockwise direction being positive.
2. (15) The coordinates of a point in one coordinate system are given, find the coordinates of that point in the coordinate system specified.

(a) $(1, -2)$ are the Cartesian coordinates $(x, y)$ of a point, find its polar coordinates.

\[ r = \sqrt{5}, \; \theta = \tan^{-1}(-2) \]

(b) $(1, 1, 2)$ are the spherical coordinates $(\rho, \theta, \phi)$ of a point, find its cylindrical coordinates.

\[ r = \rho \sin \phi = \sin 2 \approx 0.909 \]
\[ \theta = 1 \]
\[ z = \rho \cos \phi = \cos 2 \approx -0.416 \]

(c) $(1, 1, 2)$ are the Cartesian coordinates of a point in $\mathbb{R}^3$. What are its spherical coordinates?

\[ \rho = \sqrt{6} \]
\[ \theta = \tan^{-1} 1 = \frac{\pi}{4} \]
\[ \phi = \cos^{-1} \left( \frac{2}{\sqrt{6}} \right) \approx 0.615 \]
3. (30) The iterated integral \( \int_1^4 dy \int_y^{y^2} dx \) has a value equal to the area of a region \( D \) in \( \mathbb{R}^2 \).

(a) Sketch the region \( D \).

(b) Express the area of the region \( D \) by interchanging the order of integration in the given iterated integral.

\[
\int_1^4 dy \int_y^{y^2} dx = \int_1^4 dx \int_x^{x^2} dy + \int_4^{16} dx \int_{\sqrt{x}}^4 dy
\]

(c) What is the area of \( D \)?

\[
\text{area}(D) = \int_1^4 dy \int_y^{y^2} dx
\]

\[
= \int_1^4 (y^2 - y) \ dy = \frac{y^3}{3} - \frac{y^2}{2} \bigg|_1^4
\]

\[
= \left( \frac{4^3}{3} - \frac{4^2}{2} \right) - \left( \frac{1^3}{3} - \frac{1^2}{2} \right)
\]

\[
= \frac{27}{2}
\]
(d) Consider the transformation \( v = y \) and \( u = \frac{y}{\sqrt{x}} \). Let \( \hat{D} \) denote the image of \( D \) under this transformation. Sketch the region \( \hat{D} \).

\[
\begin{align*}
v & = y \\
u & = \frac{y}{\sqrt{x}}
\end{align*}
\]

(e) Compute \( \frac{\partial (x, y)}{\partial (u, v)} \)

First we write \( x \) and \( y \) as functions of \( u \) and \( v \). This gives us \( y = v \) and

\[
x = \frac{y^2}{u^2} = \frac{v^2}{u^2}.
\]

Thus, we have

\[
\frac{\partial (x, y)}{\partial (u, v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} \frac{-2v^2}{u^3} & \frac{2u}{v} \\ 0 & 1 \end{bmatrix} = \frac{-2v^2}{u^3}
\]

(f) Express \( \iint_D xy \, dA \) as an iterated integral over the region \( \hat{D} \).

\[
\iint_D xy \, dA = \int_1^4 dv \int_1^{\sqrt{v}} \frac{v^2}{u^2} \, v \left| \frac{-2v^2}{u^3} \right| \, du = \int_1^4 dv \int_1^{\sqrt{v}} \frac{2v^5}{u^3} \, du
\]
4. (20) The curve \( r = 1 + \sin \theta \), where \( r \) and \( \theta \) represent the polar coordinates of a point in \( \mathbb{R}^2 \), encloses a region \( D \) in \( \mathbb{R}^2 \).

(a) Sketch the region \( D \).

(b) Set up an integral whose value equals the area of \( D \). Note: your answer should be an iterated integral.

\[
\text{area of region} = \int_0^{2\pi} \int_0^{1+\sin \theta} r \, dr \, d\theta
\]

(c) What is the slope of the tangent line to this curve when \( \theta = 0 \)?

One way to do this is to describe the curve parametrically

\[
\Gamma(\theta) = (x(\theta), y(\theta)) = (r(\theta) \cos \theta, r(\theta) \sin \theta)
\]

\[
= ((1 + \sin \theta) \cos \theta, (1 + \sin \theta) \sin \theta)
\]

\[
= (\cos \theta + \sin \theta \cos \theta, \sin \theta + \sin^2 \theta)
\]

Then the slope of the tangent line is given by

\[
\frac{dy}{d\theta} = \frac{d}{d\theta} \left( \sin \theta + \sin^2 \theta \right)
\]

\[
= \frac{\cos \theta + \sin 2\theta}{\cos 2\theta - \sin \theta}
\]

Setting \( \theta = 0 \) in the above expression we have

\[
slope = \frac{1}{1} = 1.
\]
5. (20) Let $E$ denote the region in $\mathbb{R}^3$ that is bounded below by the plane $z = 1$ and above by the sphere $x^2 + y^2 + z^2 = 4$.

(a) If $m(x, y, z) = 1 + x^2 - z$ denotes the mass per unit volume of the region $E$. Set up an iterated integral whose value is the total mass of this region.

The projection of this region onto the $x/y$ plane is the disk centered at the origin of radius $\sqrt{3}$. It seems easiest to integrate over this region by using cylindrical coordinates. Thus,

$$ \text{total mass} = \iiint_E (1 + x^2 - z) \, dV $$

$$ = \int_0^{2\pi} \, d\theta \int_0^{\sqrt{3}} \, r \, dr \int_1^{\sqrt{4-r^2}} \left( 1 + (r \cos \theta)^2 - z \right) \, dz $$

$$ = \frac{3\pi}{10} $$

(b) Find the total surface area of this region. Note: if your answer involves an iterated integral, this integral does not have to be evaluated.

The surface consists of two separate parts. The bottom which is a disk of radius $\sqrt{3}$, and the top which is that part of the sphere lying above the $z = 1$ plane. The area of the bottom part is $3\pi$. The area of the top part is given by the integral below ($D$ denotes the disk of radius $\sqrt{3}$ centered at the origin

$$ \int \int_D \sqrt{1 + (f_x)^2 + (f_y)^2} \, dA, $$

where $f$ is the $z$ value of a point on the top half of the sphere whose equation is $x^2 + y^2 + z^2 = 4$. Differentiating implicitly we derive

$$ f_x = -\frac{x}{z} \quad \text{and} \quad f_y = -\frac{y}{z}. $$

Thus, the area of the top part is

$$ \int \int_D \sqrt{1 + (f_x)^2 + (f_y)^2} \, dA = \int \int_D \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}} \, dA $$

$$ = \int \int_D \frac{4}{z^2} \, dA $$

$$ = \int_0^{2\pi} \, d\theta \int_0^{\sqrt{3}} \frac{2r}{\sqrt{4-r^2}} \, dr. $$

So the total surface area equals

$$ 3\pi + \int_0^{2\pi} \, d\theta \int_0^{\sqrt{3}} \frac{2r}{\sqrt{4-r^2}} \, dr = 3\pi + 4\pi = 7\pi $$