1. (30) Define the following:

(a) \( \lim_{(x, y) \to (-2, 1)} f(x, y) = 5 \)

This means that for any \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that if \( 0 < \| (x, y) - (-2, 1) \| < \delta \), then \( |f(x, y) - 5| < \varepsilon \).

(b) the directional derivative of \( f \) at the point \((2, 3)\) in the direction \((1, 1)\)

Since the unit normal in the direction \((1, 1)\) is \((1/\sqrt{2}, 1/\sqrt{2})\), the directional derivative equals

\[
\lim_{h \to 0} \frac{f(2 + h/\sqrt{2}, 3 + h/\sqrt{2}) - f(2, 3)}{h}.
\]

(c) \( f(x, y) \) is differentiable at the point \((5, 7)\)

This means that there are numbers \( \epsilon_1 \) and \( \epsilon_2 \) such that

\[
f(5 + \Delta x, 7 + \Delta y) = f(5, 7) + \frac{\partial f}{\partial x}(5, 7) \Delta x + \frac{\partial f}{\partial y}(5, 7) \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,
\]

where both \( \epsilon_1 \) and \( \epsilon_2 \) approach 0 as \((\Delta x, \Delta y) \to (0, 0)\).

(d) the spherical variables \( \rho, \theta, \) and \( \phi \)

If \( P \) is a point in \( \mathbb{R}^3 \), then the spherical coordinates of \( P \) represent

\[
\rho = \text{distance of } P \text{ from the origin} = \sqrt{x^2 + y^2 + z^2}, \text{ where } (x, y, z) \text{ are the Cartesian coordinates of } P
\]

\[
\phi = \text{angle that line from origin to } P \text{ makes with the positive } z \text{ axis, } 0 \leq \phi \leq \pi
\]

\[
\theta = \text{usual polar angle. That is, the point } \hat{P} = (x, y, 0) \text{, which is the projection of } P \text{ onto the } x,y \text{ plane, has the polar coordinates } r \text{ and } \theta, \text{ where } \theta \text{ is the angle the line from the origin to the point } \hat{P} \text{ makes with the positive } x \text{ axis, with } \theta \text{ lying between } 0 \text{ and } 2\pi.
\]

(e) the flux of a force field \( F \) across a surface \( S \)

This is the surface integral of the scalar normal component of \( \vec{F} \) over the surface \( S \).
2. (25) Let \( f(x, y, z) = 2xy - z + yz^2 \).

(a) Compute the directional derivative of \( f \) at the point \((1, 1, 2)\) in the direction \( \vec{N} = (5, 0, -1) \).

The gradient of \( f \) equals \( \nabla f = (2y, 2x + z^2, 2yz - 1) \), and its value at the point \((1, 1, 2)\) is \((2, 6, 3)\). The directional derivative equals

\[
Df = \nabla f \bigg|_{(1,1,2)} \cdot \frac{(5,0,-1)}{\sqrt{26}} \\
= (2,6,3) \cdot \frac{(5,0,-1)}{\sqrt{26}} \\
= \frac{7}{\sqrt{26}}
\]

(b) Find an equation for the tangent plane to the surface \( f = 4 \) at the point \((1, 1, 2)\).

\[
(x - 1, y - 1, z - 2) \cdot (2,6,3) = 0,
\]

or

\[
2x + 6y + 3z = 14
\]

(c) What is the rate of change of \( f \) at the point \((1, 1, 2)\) along any direction tangent to the plane of part b?

Since any of these tangent directions is perpendicular to the gradient of \( f \) at that point, all of these directional derivatives will equal 0.

3. (10) A force field, \( \vec{F} \), is said to be conservative if there is a scalar valued function \( \phi \) such that \( \nabla \phi = \vec{F} \). Let \( C \) denote any path with \( P \) and \( Q \) the beginning and terminal points of the path. Explain why the line integral of the tangential component of \( F \) along the path \( C \) depends only on \( \phi \). That is, explain the formula

\[
\int_C \vec{F} \cdot d\vec{r} = \phi (Q) - \phi (P).
\]

Let \( \Gamma (t) = (x(t), y(t), z(t)) \) for \( a \leq t \leq b \) be any parametrization of the curve \( C \), such that \( \Gamma (a) = P \) and \( \Gamma (b) = Q \). Then we have

\[
\int_C \vec{F} \cdot d\vec{r} = \int_a^b (\phi_x(x(t), y(t), z(t)), \phi_y, \phi_z) \cdot (x', y', z') \ dt \\
= \int_a^b \left[ \frac{d}{dt} \phi(\Gamma (t)) \right] dt = \phi (\Gamma (b)) - \phi (\Gamma (a)) \\
= \phi (Q) - \phi (P).
\]
4. (25) Let \( S \) be the rectangular region in the \( x, z \) plane that is bounded by the lines \( x = 2, \) \( x = 0, \) \( z = 0, \) \( z = 1 \). Let \( \vec{F} = (xy - z, x + 2z, \cos xy) \).

(a) \( \text{curl} \vec{F} = \)

\[
\text{curl} \vec{F} = \det \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\partial_x & \partial_y & \partial_z \\
xy - z & x + 2z & \cos xy
\end{vmatrix}
\]

\[
= (-x \sin xy - 2, -1 + y \sin xy, 1 - x)
\]

(b) Compute the flux of \( \text{curl} \vec{F} \) crossing the surface \( S \) in the direction of increasing \( y \) directly from the definition of flux as a surface integral.

\[
\iint_S \text{curl} \vec{F} \cdot dS = \iint_S (-2, -1, 1 - x) \cdot (0, 1, 0) \ dS
\]

\[
= \iint_S (-1) \ dS = -\text{area} (S)
\]

\[
= -2
\]

(c) Compute the flux of \( \text{curl} \vec{F} \) crossing the surface \( S \) in the direction of increasing \( y \) by using Stoke’s theorem.

Stoke’s theorem says that \( \iiint_S \text{curl} \vec{F} \cdot dS = \oint_{\partial S} \vec{F} \cdot d\vec{r} \), where the path that is the boundary of \( S \) is traced out in a direction compatible with the normal direction used in computing the surface integral. To use Stokes theorem in this problem we note that the normal direction is in the positive \( y \) direction so if we look at \( S \) from the positive \( y \) axis the direction of integration around the boundary of \( S \) must be in the counter clockwise direction.

Let \( C_1 \) represent that part of the boundary of \( S \) for which \( z = 0, \) \( C_2 \) for \( x = 0, \) \( C_3 \) for \( z = 1, \) and \( C_4 \) for \( x = 2. \) Then we have

\[
\iint_S \text{curl} \vec{F} \cdot dS = \oint_{\partial S} \vec{F} \cdot d\vec{r} = \oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{C_2} \vec{F} \cdot d\vec{r} + \oint_{C_3} \vec{F} \cdot d\vec{r} + \oint_{C_4} \vec{F} \cdot d\vec{r}
\]

\[
= \int_2^0 (0, x, 1) \cdot (dx, 0, 0) + \int_0^1 (-z, 2z, 1) \cdot (0, 0, dz)
\]

\[
+ \int_0^2 (-1, x + 2, 1) \cdot (dx, 0, 0) + \int_1^0 (-z, 2 + 2z, 1) \cdot (0, 0, dz)
\]

\[
= \int_2^0 dx + \int_0^1 dz + \int_0^2 (-1) dx + \int_1^0 dz
\]

\[
= -2
\]
5. (20) Let $C$ denote the curve that is the intersection of the surfaces $x^2 + y^2 + z = 9$ and $z = 5$. Let $f(x, y, z) = x + y - 2z$.

(a) Find all critical points of $f$.

The critical points of $f$ are those points where the gradient of $f$ either does not exist or it equals $\mathbf{0}$. Since $f$ is a polynomial its gradient exists everywhere, so $(x, y, z)$ is a critical point of $f$ only if $\nabla f = \mathbf{0}$ at that point.

$$\nabla f = (1, 1, -2)$$

This can never be zero so $f$ has no critical points.

(b) Find the maximum value that the function $f$ attains on the curve $C$.

Use the Lagrange multiplier technique and find a solution to the equations

$$\nabla f + \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 = (0, 0, 0)$$

$$g_1 = x^2 + y^2 + z = 9$$
$$g_2 = z = 5.$$ 

The first equation is equivalent to the following equations

$$1 + 2\lambda_1 x = 0$$
$$1 + 2\lambda_1 y = 0$$
$$-2 + \lambda_1 + \lambda_2 = 0.$$ 

Thus, we have $x = \frac{-1}{2\lambda_1} = y$. Since $x, y, z$ must satisfy $g_1 = 0$, and $z = 5$, $\lambda_1$ satisfies

$$\frac{1}{4\lambda_1^2} + \frac{1}{4\lambda_1^2} = 4$$
$$\frac{1}{4\lambda_1^2} = 2$$
$$\frac{1}{2\lambda_1} = \pm \sqrt{2}$$

So we have $x = y = \pm \sqrt{2}$. Thus, the maximum value of $f$ on the curve $C$ occurs at $(\sqrt{2}, \sqrt{2}, 5)$ or at $(-\sqrt{2}, -\sqrt{2}, 5)$, and

$$f \left(\sqrt{2}, \sqrt{2}, 5\right) = 2\sqrt{2} - 10$$
$$f \left(-\sqrt{2}, -\sqrt{2}, 5\right) = -2\sqrt{2} - 10$$

Hence the maximum value of $f$ on the curve $C$ is $2\sqrt{2} - 10$. 

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6. (40) Let $S$ denote the surface that encloses the region $E$, in $\mathbb{R}^3$, which is bounded by $x = 0$, $y = 0$, $z = 0$, and $z = 9 - x^2 - y^2$. Let $S_1$, $S_2$, and $S_3$ denote those parts of $S$ that lie in the planes $x = 0$, $y = 0$, and $z = 0$ respectively, and let $S_4$ denote the remaining part of $S$. Let $\vec{F}(x, y, z) = (x, y, -z)$.

The region $E$ is shown below:

(a) Find the volume of $E$.

$$
\text{volume } (E) = \iiint_E dV = \int_0^{\pi/2} \int_0^3 r \, dr \int_0^{9-r^2} dz = \frac{\pi}{2} \int_0^3 (9r - r^3) \, dr = \frac{\pi}{2} \left( \frac{9r^2 - r^4}{4} \right) \bigg|_0^3 = 81\pi \frac{8}{8}
$$

(b) Find the area of $S_4$.

The surface $S_4$ is the graph of the function $f(x, y) = 9 - x^2 - y^2$ for $x^2 + y^2 \leq 9$. Thus, the area of $S_4$ equals

$$\text{area } (S_4) = \iint_D \sqrt{1 + 4x^2 + 4y^2} \, dA = \int_0^{\pi/2} \int_0^3 r \sqrt{1 + 4r^2} \, dr = \frac{\pi}{2} \left( \frac{37\sqrt{37} - 1}{12} \right)$$
(c) Find the outward flux of $\mathbf{F}$ across each of the surfaces $S_1$, $S_2$, and $S_3$.

The flux of $\mathbf{F}$ across these surfaces equals

$$\text{flux} (S_1: x = 0) = \int \int_{S_1} \mathbf{F} \cdot d\mathbf{S} = \int \int_{S_1} (0, y, -z) \cdot (-1, 0, 0) \ dS = 0$$

$$\text{flux} (S_2: y = 0) = \int \int_{S_2} \mathbf{F} \cdot d\mathbf{S} = \int \int_{S_1} (x, 0, -z) \cdot (0, -1, 0) \ dS = 0$$

$$\text{flux} (S_3: z = 0) = \int \int_{S_3} \mathbf{F} \cdot d\mathbf{S} = \int \int_{S_1} (x, y, 0) \cdot (0, 0, -1) \ dS = 0$$

(d) Use the divergence theorem to find the outward flux of $\mathbf{F}$ across the surface $S$, and deduce what the outward flux of $\mathbf{F}$ across $S_4$ must equal.

The total outward flux across the surface $S$ equal

$$\text{flux} (S) = \int \int_{S} \mathbf{F} \cdot d\mathbf{S} = \int \int \int_{E} \text{div} \left( \mathbf{F} \right) \ dV$$

$$= \int \int \int_{E} dV = \text{volume} (E) = \frac{81\pi}{8}.$$

The outward flux across the surface $S_4$ plus the sums of the outward fluxes across all of the other surfaces making up the surface $S$ must equal the total outward flux. Thus,

$$\text{flux} (S_4) = \text{flux} (S) - \text{flux} (S_1) - \text{flux} (S_2) - \text{flux} (S_3)$$

$$= \frac{81\pi}{8}$$