This exam has two parts. Part A must be done in class; part B is a take home, and is due at the beginning of class on Wednesday March 25. Late work will not be accepted. You are welcome to discuss the test with me at anytime. However, you are not to talk with anyone else about the problems on this exam. All work you wish to have graded must be in your bluebooks.

Part A

1. (15) Consider the differential equation \( y' = 6y + t \).

(a) What adjectives would you use to describe this equation?

**Ans:** first order, linear, non-homogeneous, non-autonomous

(b) Describe how you would find a solution, using pencil and paper, to this differential equation.

**Ans:** I would first rewrite it in the form \( y' - 6y = t \). Then I would multiply this last equation by \( e^{-6t} \). This, after some simplification, leads to the equation \( \frac{d}{dt}(e^{-6t}y) = te^{-6t} \). I would then integrate both sides of this equation, and solve for \( y \).

(c) Using the method you described in part b., find the solution \( y(t) \) to this equation which also satisfies \( y(0) = 0 \).

**Ans:** Following the process above we get \( y(t) = ce^{6t} - \frac{1}{6}t - \frac{1}{36} \). Setting \( y(0) = 0 \) and solving for \( c \), we have \( c = \frac{1}{36} \). Thus, the solution to the initial value problem is:

\[
y(t) = \frac{e^{6t} - 1}{36} - \frac{t}{6}.
\]

2. (10) Suppose the function \( x(t) \) satisfies the differential equation \( x' = (x - 1) \sin t \) and \( x(2.6) = 1 \). What must \( x(0) \) equal? Be sure to explain why must it equal this value?

**Ans:** Notice that \( x = 1 \) is an equilibrium solution to this differential equation, and that the function \( (x - 1) \sin t \) and both of its partial derivative are continuous everywhere. Thus, the existence-uniqueness theorem tells us that if a solution ever takes on the value 1, then it (the solution) is identically 1. In otherwords \( x(0) = 1 \).
3. (20) The following system models a predator/prey relationship between two species.

\[
\frac{dx}{dt} = -x + 6xy \\
\frac{dy}{dt} = 2y - 8y^2 - 20xy
\]

(a) Which variable represents the predator and why?

**Ans:** The predators are represented by the \( x \) variable. The equation describing the rate of change of \( x \) with respect to \( t \) indicates that, with no interaction between the two species, the \( x \)-species will die out. More importantly, any interaction between the two species has a positive effect on \( x \)'s growth rate, and a negative effect on \( y \)'s growth rate.

(b) If \( x = 0 \), then we have a single differential equation for \( y \). Draw the phase line for this single equation and sketch the graphs of solutions to this single equation for various initial conditions.

(c) Find the equilibrium points for the system.

**Ans:** The equilibrium points are: \((0,0)\), \((0,1/4)\), \((1/30,1/6)\).

(d) Sketch the direction field for the system.

**Ans:** The first plot is the direction field as given by Maple; the second plot shows the nullclines which intersect at the equilibrium point \((1/30,1/6)\). Notice the rotational movement about this equilibrium point.
(e) What behavior would you expect solutions, which start close to an equilibrium point of the system, to exhibit?

**Ans:** At the equilibrium point (0, 0) most solutions tend away from the point, except for those solutions which start on the line \( y = 0 \). Solutions which start on this line stay on this line and tend to the origin as \( t \to \infty \). This equilibrium point looks like a saddle point.

At the equilibrium point (0, 1/4) most solutions tend away from the point, except for a line of solutions which tend to the (0, 1/4) as \( t \to \infty \). This equilibrium point looks like a saddle point too.

The equilibrium point (1/30, 1/6) acts like a spiral sink. That is solutions which start out near this point spiral around this point and converge to it as \( t \to \infty \).

The easiest way to see the behavior of solutions at an equilibrium point is to examine the direction field of this system in a small rectangle about that point.

4. (20) The following questions refer to the differential equation

\[
\frac{dx}{dt} = f(\alpha, x) = x^2 + 2x + \alpha^2.
\]

(a) Draw the bifurcation diagram for this one parameter family of differential equations.

**Ans:**

![Bifurcation Diagram](image)

The equation \( x^2 + 2x + \alpha^2 = 0 \) (complete the square) can be written as \( (x + 1)^2 + \alpha^2 = 1 \). This latter form is easily seen to describe a circle of radius 1 centered at the point \((0, -1)\).

(b) Locate all bifurcation values.

**Ans:** As is clear from the bifurcation diagram or from the equation which gives \( x \) as a function of \( \alpha \) \((x_\alpha = -1 \pm \sqrt{1 - \alpha^2})\), the bifurcation values of \( \alpha \) are \( \alpha = \pm 1 \). If \( \alpha \) is less than \(-1\) or greater than 1, the equation has no equilibrium points. If \( \alpha = 1 \) or if \( \alpha = -1 \), the equation has exactly one equilibrium point. If \( \alpha \) lies between \(-1\) and 1, then the equation has two equilibrium points.
(c) Determine which parts of the bifurcation diagram consist of sinks, sources, and nodes.

**Ans:** For each value of \( \alpha \) the numbers \( x_\alpha = -1 \pm \sqrt{1 - \alpha^2} \) are equilibrium points. The partial derivative of the right hand side of the differential equation with respect to \( x \) equals \( 2x + 2 \). Evaluating this at the equilibrium points, we have:

\[
\frac{\partial f}{\partial x} (\alpha, x_\alpha) = 2 \left( -1 \pm \sqrt{1 - \alpha^2} \right) + 2 = \pm 2 \sqrt{1 - \alpha^2}.
\]

This expression is non-zero for \( \alpha \) strictly between \(-1\) and \(1\). It is positive for these \( \alpha \) when the plus sign is present, i.e., on the upper half of the bifurcation curve, and negative on the lower half of the bifurcation curve. Thus, the top half of the curve consists of sources and the bottom half consists of sinks. At the bifurcation values, \( \alpha = \pm 1 \), the differential equation is \( x' = (x+1)^2 \). From this we see that the equilibrium point \( x = -1 \) is a node for \( \alpha = -1 \) and \( \alpha = 1 \).

(d) If \( x(t) \) satisfies \( x' = f(2,x) \) and \( x(0) = 0 \), what is the limit of \( x(t) \) as \( t \to \infty \)?

**Ans:** For all \( \alpha \) greater than \( 1 \), the right hand side of the differential equation is positive. Thus, for \( \alpha = 2 \), any solution of this differential equation will be an increasing function, and if the domain of the function consists of all positive numbers, then the limit as \( t \to \infty \) of the solution will be infinite. However, it is easy to see that solutions of the differential equation \( x' = x^2 + 2x + 4 \) are

\[
x(t) = \sqrt{3} \tan \left( \sqrt{3}t + c \right) - 1.
\]

Thus, we see that these solutions are only defined on a finite interval, and that the solutions become infinite as \( t \) approaches the right endpoint of the interval.

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**Part B**

5. (20) For the second-order, homogeneous, constant coefficient, differential equation, where both \( p \) and \( q \) are real numbers,

\[
\frac{d^2 y}{dt^2} + p \frac{dy}{dt} + q y = 0:
\]

(a) Write this as a first-order system.

**Ans:** If we let \( x_1 = y \) and \( x_2 = y' \), then these functions satisfy the following system of differential equations.

\[
\frac{dx_1}{dt} = x_2
\]
\[
\frac{dx_2}{dt} = -qx_1 - px_2.
\]
(b) What is the condition on $p$ and $q$ which guarantee that the eigenvalues of the corresponding system are complex?

**Ans:** $p^2 - 4q < 0$.

(c) What relationship between $p$ and $q$ guarantees that the origin is a spiral sink? What relationship guarantees that the origin is a spiral source? What relationship guarantees that the origin is a center?

**Ans:** The eigenvalues of the coefficient matrix of this system are $\lambda = \frac{-p}{2} \pm \frac{1}{2} \sqrt{p^2 - 4q}$. To have a spiral behavior we need the eigenvalues to be complex. This is the case when $p^2 - 4q < 0$. So assuming this condition is met:

- spiral sink: $p > 0$
- spiral source: $p < 0$
- center: $p = 0$

(d) If the eigenvalues are complex, what condition on $p$ and $q$ guarantees that the solutions will spiral around the origin in a clockwise direction?

**Ans:** To ensure that solutions spiral around the origin we need $p^2 - 4q < 0$, and to ensure that the rotations are in a clockwise direction we need $q > 0$. An easy way to see this is to look at the nullclines in the various cases for $p$ and $q$. A second way is to show that the polar angle $\theta(t)$ is a decreasing function of $t$. See the figure below.

![Diagram](image)

\[ \tan(\theta(t)) = \frac{x_2}{x_1} \]

The following calculations show that if $q > 0$ than the rotation is counterclockwise.

\[
\begin{align*}
\sec^2(\theta(t)) \frac{d\theta}{dt} &= \frac{d}{dt} \tan(\theta(t)) = \frac{d}{dt} \left( \frac{x_2}{x_1} \right) \\
&= \frac{x_2' x_1 - x_2 x_1'}{x_1^2} \\
&= \frac{(-qx_1 - px_2)x_1 - x_2^2}{x_1^2} = \ldots \\
&= -\frac{q}{x_1} \left( x_1 + \frac{p}{2q} x_2 \right)^2 + \frac{x_2^2}{4q^2} (4q - p^2) \\
\end{align*}
\]

Thus, we see that the sign of $\frac{d\theta}{dt}$ is negative if $q > 0$. We are also assuming that $4q - p^2 > 0$ in order to ensure that we do have rotational motion.
6. (20) Suppose you know that \( \lambda_1 = 2 \) and \( \lambda_2 = -4 \) are the eigenvalues of the coefficient matrix of a two dimensional linear homogeneous system of differential equations. Suppose they have \( \xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) and \( \xi_2 = \begin{bmatrix} -4 \\ 1 \end{bmatrix} \) as their respective eigenvalues.

(a) Sketch the direction field for this system. Put in as much information as you can, i.e., straight line solutions, asymptotic behavior of solutions.

\[
\text{Ans:}
\]

\[
\begin{align*}
\lambda_1 &= 2 \\
\lambda_2 &= -4 \\
\xi_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\xi_2 &= \begin{bmatrix} -4 \\ 1 \end{bmatrix}
\end{align*}
\]

(b) Find the solution to this system which satisfies the initial conditions \( \tilde{X}(0) = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \).

\[
\text{Ans:}
\]

\[
\tilde{X}(t) = \frac{-6}{5} e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{-4}{5} e^{-4t} \begin{bmatrix} -4 \\ 1 \end{bmatrix}
\]

7. (20) Suppose a factory has two vats. Vat number 1 which contains 100 gallons of a salt solution (1/8 pound per gallon) and vat number 2 which contains 50 gallons of a sugar solution (1/2 pound per gallon). Suppose the two tanks are connected and two gallons per minute flows from vat number 1 into vat number 2 and at the same time two gallons per minute flows from vat number 2 into vat number 1.

(a) Find a system of differential equations which models the number of pounds of salt and sugar in each vat. (You will probably want to consider a four dimensional system.) Be sure to include the initial conditions.

\[
\text{Ans:}
\]

Let

\[
\begin{align*}
x_1(t) &= \text{the number of pounds of salt in vat 1 at time } t. \\
x_2(t) &= \text{the number of pounds of salt in vat 2 at time } t. \\
x_3(t) &= \text{the number of pounds of sugar in vat 1 at time } t. \\
x_4(t) &= \text{the number of pounds of sugar in vat 2 at time } t.
\end{align*}
\]
Then we have

\[
\begin{align*}
\frac{dx_1}{dt} &= 2 \left( \frac{x_2}{50} \right) - 2 \left( \frac{x_1}{100} \right) \quad x_1(0) = \frac{25}{2} \\
\frac{dx_2}{dt} &= 2 \left( \frac{x_1}{100} \right) - 2 \left( \frac{x_2}{50} \right) \quad x_2(0) = 0 \\
\frac{dx_3}{dt} &= 2 \left( \frac{x_4}{50} \right) - 2 \left( \frac{x_3}{100} \right) \quad x_3(0) = 0 \\
\frac{dx_4}{dt} &= 2 \left( \frac{x_3}{100} \right) - 2 \left( \frac{x_4}{50} \right) \quad x_4(0) = 25
\end{align*}
\]

Note: if we add the first two equations together we get \( \frac{d(x_1 + x_2)}{dt} = 0 \), which means that the total amount of salt in the two vats is constant. A similar remark is true for the total amount of sugar.

(b) Find the solution to part a. That is find functions which give the pounds of salt in each vat at time \( t \), and functions which give the pounds of sugar in each vat at time \( t \).

Ans:

\[
\begin{align*}
x_1(t) &= \frac{25}{6} e^{-3t/50} + \frac{25}{3} \\
x_2(t) &= \frac{25}{6} - \frac{25}{6} e^{-3t/50} \\
x_3(t) &= \frac{50}{3} - \frac{50}{3} e^{-3t/50} \\
x_4(t) &= \frac{50}{3} e^{-3t/50} + \frac{25}{3}
\end{align*}
\]

(c) What is the limiting value as \( t \to \infty \) of the sugar and salt in each vat?

Ans: The limiting values of salt and sugar in each vat are: \( 25/3 \) pounds of salt and \( 50/3 \) pounds of sugar in the first vat and the second vat has \( 25/6 \) pounds of salt and \( 25/3 \) pounds of sugar.