Each problem is worth 20 points. You must work the first two problems, and then any three of problems 3 through 7.

1. Define the following terms and give an example of each. **No** example, **no** credit.

(a) Uniformly continuous.

**Ans:** A function \( f \) is uniformly continuous on a set \( E \) if for every \( \epsilon > 0 \), there is a \( \delta > 0 \) such that if \( x, y \in E \) and \( |x - y| < \delta \), then \( |f(x) - f(y)| < \epsilon \). The function \( f(x) = x \) is uniformly continuous on the entire real line.

(b) Sequentially compact.

**Ans:** A set \( E \) is sequentially compact if every sequence of points which belongs to the set \( E \) has a subsequence which converges to a point in \( E \). The set which consists of the single point \( x = 1 \) is sequentially compact.

(c) Cluster point.

**Ans:** A point \( x_0 \) is a cluster point of a set \( E \) if for every \( \delta > 0 \), the open interval \((x_0 - \delta, x_0 + \delta)\) contains an infinite number of points of \( E \). If \( E = \{1/n : n \in \mathbb{N}\} \), then 0 is a cluster point of \( E \).

(d) \( \lim_{x \to x_0} f(x) = L \).

**Ans:** \( x_0 \) is assumed to be a cluster point of the set \( E \). If both \( x_0 \) and \( L \) are finite the limit is defined as \( \lim_{x \to x_0} f(x) = L \), if for every \( \epsilon > 0 \), there is a \( \delta > 0 \) such that if \( x \in E \) and \( 0 < |x - x_0| < \delta \), then \( |f(x) - L| < \epsilon \).

If \( x_0 \) is finite and \( L \) is infinite we define the limit as follows: For every \( M \) there is a \( \delta \) such that if \( x \in E \) and \( 0 < |x - x_0| < \delta \), then \( f(x) > M \).

If both \( x_0 \) and \( E \) are infinite we define the limit as follows. For every \( M \) there is an \( N \) such that if \( x \in E \) and \( x > N \), then \( f(x) > M \).

Definitions for other possibilities of \( x_0 \) and \( L \) are similar.
2. Prove any two of the following theorems. Be sure to say which two you want graded.

(a) If \( f(x) \) is continuous on a closed bounded interval \([a, b]\), then there exists a point \( x_0 \in [a, b] \) such that \( f(x_0) = \sup \{ f(x) : a \leq x \leq b \} \).

**Ans:** Assume we have shown that if \( f \) is continuous on a closed bounded interval, then the range of \( f \) is also bounded. Let \( M = \sup \{ f(x) : a \leq x \leq b \} \). Then there is a sequence of points \( \{x_n\}_{n=1}^{\infty} \subseteq [a, b] \), such that \( \lim_{n \to \infty} f(x_n) = M \). Since the sequence \( \{x_n\}_{n=1}^{\infty} \) is bounded it possesses a convergent subsequence \( \{x_{n_i}\}_{i=1}^{\infty} \), and this subsequence converges to some point \( x_\infty \in [a, b] \). The continuity of \( f \) on the interval implies that \( M = \lim_{i \to \infty} f(x_{n_i}) = f(x_\infty) \).

(b) Show that \( x_0 \) is a cluster point of a set \( E \) if and only if there is a sequence of distinct points \( \{a_n\}_{n=1}^{\infty} \) of \( E \) such that \( \lim_{n \to \infty} a_n = x_0 \).

**Ans:** Suppose that \( x_0 \) is a cluster point of \( E \). Then we can inductively construct a sequence of distinct points which converges to \( E \). Since \( x_0 \) is a cluster point of \( E \) there is a point \( a_1 \in E \) such that \( |a_1 - x_0| < 1 \) and \( a_1 \neq x_0 \). Remember every open interval about \( x_0 \) must contain an infinite number of points of \( E \). Assume we have picked the first \( n \) points of our sequence and they satisfy \( |a_i - x_0| < 1/i \), they are distinct from one-another, and none of them equals \( x_0 \). Let \( \delta = \min \{ \frac{1}{n+1}, |a_i - x_0| \text{ for } 1 \leq i \leq n \} \).

Then \( \delta > 0 \). Pick \( a_{n+1} \) from \( E \) such that \( a_{n+1} \neq x_0 \) and \( |a_{n+1} - x_0| < \delta \leq \frac{1}{n+1} \). Thus, we have shown there is a sequence of distinct points of \( E \) which converges to the cluster point \( x_0 \).

Suppose next that \( x_0 \) is a point which is the limit of a sequence \( \{a_n\}_{n=1}^{\infty} \) of distinct points of \( E \). Let \( \delta > 0 \). Then there is an \( N \) such that if \( n > N \), then \( |a_n - x_0| < \delta \). Thus, the open interval \((x_0 - \delta, x_0 + \delta)\) contains all of the points \( x_n \) for \( n > N \). Hence this interval contains an infinite number of points of \( E \). Remember the sequence \( \{a_n\}_{n=1}^{\infty} \) consists of distinct points.
(c) Let \( f(x) \) be an increasing function on the open interval \((a, b)\).
Show that \( \lim_{x \to x_0^+} f(x) = \inf\{ f(x) : x_0 < x \} \) for any \( x_0 \in (a, b) \).

**Ans:** Let \( L = \inf\{ f(x) : x_0 < x \} \). To see that \( L = \lim_{x \to x_0^+} f(x) \), let \( \epsilon > 0 \). Then there exists an \( x_1 > x_0 \) such that \( f(x_1) < L + \epsilon \). Set \( \delta = x_1 - x_0 \). Then if \( x_0 < x < x_0 + \delta \), i.e., \( x_0 < x < x_1 \), we have \( L \leq f(x) \leq f(x_1) < L + \epsilon \). Which means that \( \lim_{x \to x_0^+} f(x) = L \).

Remember: you need work only three of the remaining problems.

3. Let \( f(x) \) be differentiable on \((0, \infty)\) and suppose that \( L = \lim_{x \to \infty} f'(x) \) exists and is finite. Prove that if \( \lim_{x \to \infty} f(x) \) exists and is finite, then \( L = 0 \).

**Ans:** The fact that \( L \) must equal zero under these conditions follows from the Mean Value Theorem. For each \( n \in \mathbb{N} \) we have

\[
 f(n + 1) - f(n) = f'(c_n),
\]

where \( c_n \) lies between \( n \) and \( n + 1 \). The limit of the left hand side as \( x \to \infty \) is 0. Thus, we have a sequence of points \( c_n \) which tends to infinity and for which \( \lim_{n \to \infty} f'(c_n) = 0 \). However, we know that the limit of \( L = \lim_{x \to \infty} f'(x) \) exists. Thus, this limit must equal the limiting value of \( f'(c_n) = 0 \).

4. Evaluate the following limits.

(a) \( \lim_{x \to 1} \frac{\ln x}{\sin(\pi x)} \).

**Ans:** \( \lim_{x \to 1} \frac{\ln x}{\sin(\pi x)} = \lim_{x \to 1} \frac{1/x}{\pi \cos(\pi x)} = -\frac{1}{\pi} \).

(b) \( \lim_{x \to 0^+} \frac{\cos x - e^x}{\ln(1 + x^2)} \).

**Ans:** \( \lim_{x \to 0^+} \frac{\cos x - e^x}{\ln(1 + x^2)} = \lim_{x \to 0^+} \frac{-\sin x - e^x}{\ln(1 + x^2)} \lim_{x \to 0^+} \frac{(1 + x^2)(\sin x + e^x)}{2x} = -\infty. \)
(c) \( \lim_{x \to 0} \left( \frac{x}{\sin x} \right)^{1/x^2} \).

\[ \text{Ans: } \lim_{x \to 0} \left( \frac{x}{\sin x} \right)^{1/x^2} = \lim_{x \to 0} e^{\frac{\ln(x/\sin x)}{x^2}}. \]

Taking the limit of the exponent as \( x \to \infty \), we have

\[
\lim_{x \to 0} \frac{\ln(x/\sin x)}{x^2} = \lim_{x \to 0} \frac{\ln x - \ln \sin x}{x^2} = \lim_{x \to 0} \frac{1/x - \cos x/\sin x}{2x} = \lim_{x \to 0} \frac{\sin x - x \cos x}{2x^2 \sin x} = \lim_{x \to 0} \frac{x \sin x}{4x \sin x + 2x^2 \cos x} = \lim_{x \to 0} \frac{2 \cos x - x \sin x}{4 \sin x + 8x \cos x - 2x^2 \sin x} = \lim_{x \to 0} \frac{12 \cos x - 12x \sin x - 2x^2 \cos x}{12 \cos x - 12x \sin x - 2x^2 \cos x} = 1/6.
\]

Thus, \( \lim_{x \to 0} \left( \frac{x}{\sin x} \right)^{1/x^2} = e^{1/6} \).
5. A set of real numbers is said to be closed if it contains all of its cluster points. Show that a set which is both closed and bounded is sequentially compact.

**Ans:** Let \( E \) be any set which is closed and bounded. Suppose that \( \{a_n\}_{n=1}^{\infty} \) is any sequence of points in \( E \). If the sequence contains only a finite collection of points of \( E \), then it has a subsequence which consists of only one point. This subsequence converges to this point which belongs to \( E \). If the original sequence contains an infinite number of distinct points of \( E \), let \( \{b_n\}_{n=1}^{\infty} \) be a subsequence all of whose points are distinct. Then, since \( E \) is bounded this subsequence is bounded. By the Bolzano-Weierstrass theorem it has a convergent subsequence. Since this subsequence consists of distinct points, the value it converges to is a cluster point of \( E \). Since \( E \) is closed this number belongs to \( E \). Thus, the original sequence contains a subsequence which converges to a point of \( E \). Hence \( E \) is sequentially compact.

6. Suppose that \( f(x) \) is uniformly continuous on the open interval \((0,1)\).

(a) Let \( \{a_n\}_{n=1}^{\infty} \subset (0,1) \), be a Cauchy sequence. Show that \( \{f(a_n)\}_{n=1}^{\infty} \) is a Cauchy sequence.

**Ans:** To see that the sequence \( \{f(a_n)\}_{n=1}^{\infty} \) is a Cauchy sequence, let \( \epsilon > 0 \). Then there is a \( \delta > 0 \) such that if \( |x-y| < \delta \), then \( |f(x)-f(y)| < \epsilon \). For this \( \delta \) there is an \( N \) such that if \( n, m > N \) then \( |a_n-a_m| < \delta \). Thus, for this \( N \) we have \( |f(a_n)-f(a_m)| < \epsilon \).

(b) Show that \( \lim_{x \to 0^+} f(x) \) exists and is finite.

**Ans:** Let \( \{x_n\} \) be any sequence in \((0,1)\) which converges to 0. This sequence is a Cauchy sequence and by the previous problem the sequence \( \{f(x_n)\} \) is also Cauchy. Since every Cauchy sequence converges, this last sequence has a limit. Call it \( y_0 \) and define \( f(0) = y_0 \). Note that \( y_0 \) must be a finite number. We need to show that \( \lim_{x \to 0^+} f(x) = y_0 \). It will suffice to show that for any sequence \( \{a_n\} \subset (0,1) \) which converges to 0, that \( f(a_n) \) converges to \( y_0 \). So let \( \{a_n\} \) be such a sequence. Form the new sequence \( c_n = x_n \) if \( n \) is odd and \( c_n = a_n \) if \( n \) is even. Then \( c_n \) converges to zero and is Cauchy. Thus, \( f(c_n) \) is also a Cauchy sequence and converges to some value \( z \). However, the subsequence \( f(x_n) \) converges to \( y_0 \). Thus, \( z = y_0 \). Moreover, the subsequence \( f(a_n) \) is also Cauchy and must converge. So it too must converge to \( z = y_0 \). Thus, for every sequence \( a_n \) in \((0,1)\) which converges to 0, \( f(a_n) \) converges to \( y_0 \).
7. Let \( f(x) = x \ln x \) for \( 0 < x \).

(a) Graph this function, be sure to explain why your plot looks like it does.

**Ans:** Before plotting this function, we gather some facts about it. First we need to decide what its limiting behavior is at \( x = 0 \). Thus, \( \lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = 0 \). Thus, the function remains bounded on every bounded interval of positive real numbers. Its limit as \( x \to 1 \) is of course infinite. To find out where this function has extreme values, if any, we compute its derivative. \( \frac{d}{dx}(x \ln x) = \ln x + 1 \).

Since \( \ln x \) is a strictly increasing function on \((0, \infty)\) and takes on every real number exactly once, the derivative is negative for \( x < e^{-1} \), since the derivative is zero at this value. For \( x > e^{-1} \) the derivative is positive. Thus, \( x \ln x \) decreases from 0 at \( x = 0 \) to \(-e^{-1}\) at \( x = e^{-1} \) and it then increases for all larger values of \( x \). A plot of the function is shown below.

(b) Determine all intervals on which this function is monotone.

**Ans:** From the above discussion of the derivative of \( x \ln x \) we see that \( x \ln x \) is decreasing on the interval \((0, e^{-1})\) and increasing on the interval \((e^{-1}, \infty)\).