1. (10) Given a collection of stock prices \( S_i \) we calculate the price change factors \( u \) and \( d \) as follows

\[
d = 1 + \sqrt{\xi t} \\
u = 1 + \sqrt{\xi t}
\]

where \( \xi t \) represents the time interval between the \( S_i \) and is the same for each \( i \). The question arises what happens if we use a different scale to measure the same unit of time. For example if \( \xi t \) is to represent one day then depending upon what our basic unit of time is the numerical value of \( \xi t \) could be 1, \( \frac{1}{365} \), \( \frac{1}{5} \), etc. Suppose we have two different time scales \( T_1 \) and \( T_2 \), where \( T_2 = \frac{1}{T_1} \). For example if \( T_2 \) is years and \( T_1 \) is days, then one year equals 365 days, or maybe 360 days, or perhaps the number of business days in a year.

a. Let \( \xi t_0 \) represent the actual amount of time between the different stock prices. Show that the \( u \) and \( d \) for this time step are the same regardless of what unit of time is used.

Let \( \xi t_{P_1} \) and \( \xi t_{P_2} \) be the value assigned to \( \xi t_0 \) in the two different time scales. Let \( M : \frac{\xi t_{P_2}}{N} \geq \frac{\xi t_{P_1}}{N} \) and

\[
S^2 : \frac{\xi t_{P_2}}{N} \geq \frac{\xi t_{P_1}}{N}
\]

Let \( u_i \) and \( d_i \) denote the jumps as calculated in the two different time scales. Then

\[
\begin{align*}
u_1 : 1 + \frac{M}{\sqrt{\xi t_{P_1}}} + \frac{S}{\sqrt{\xi t_{P_1}}} : M + S : \sqrt{\xi t_{P_1}} : u_2 \\
d_1 : 1 + \frac{M}{\sqrt{\xi t_{P_1}}} \frac{S}{\sqrt{\xi t_{P_1}}} : M : S : \sqrt{\xi t_{P_1}} : d_2
\end{align*}
\]

b. Suppose you wanted to calculate a \( u \) and \( d \) for a time step, which is different from the time step given by the data. Show that the \( u \) and \( d \) values you calculate do not depend upon the time scale chosen.

Let \( \xi t_{P_0} \) denote the new time step, and let \( \xi t_{P_1} \) and \( \xi t_{P_2} \) be the values assigned to this new time step in the different units of time. Then there is a number \( G \) such that \( \xi t_{P_0} = \xi t_{P_1} \). Let \( u_i \) and \( d_i \) denote the values of the price change factors for this new time step. Then

\[
\begin{align*}
u_1 : 1 + \frac{M}{\sqrt{\xi t_{P_1}}} + \frac{S}{\sqrt{\xi t_{P_1}}} : 1 + \sqrt{\xi t_{P_1}} : u_2 \\
d_1 : 1 + \frac{M}{\sqrt{\xi t_{P_1}}} \frac{S}{\sqrt{\xi t_{P_1}}} : 1 + \sqrt{\xi t_{P_1}} : d_2
\end{align*}
\]
2. (10) We’ve seen that if we have three European call options on the same stock with the same expiration date and $X_1 < X_2 < X_3$, with $X_2 = \gamma X_1 + \gamma X_3$, then $C_2 = C_1 + \gamma X_1 < c P_3$. Find and prove a similar formula for European put options.

Let $A_1$ denote a portfolio which consists of the second put option, and let $A_2$ denote the portfolio, which consists of units of the first put option and $\gamma$ units of the third put option. The table below shows the values of these two portfolios at expiration as a function of the stock price $S$.

<table>
<thead>
<tr>
<th></th>
<th>$S &lt; X_1$</th>
<th>$X_1 &lt; S &lt; X_2$</th>
<th>$X_2 &lt; S &lt; X_3$</th>
<th>$X_3 &lt; S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$X_2 - S$</td>
<td>$X_2 - S$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$c \gamma X_1 + \gamma X_1 - S$</td>
<td>$c \gamma X_1 + \gamma X_1 - S$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

It is clear that the value of the second portfolio is at least as much as the value of the second portfolio in the cases where $X_2 < S$. (Since $X_1 < X_2 < X_3$, $c$ satisfies $0 < c < 1$.) To see that this is also true in the first two cases we express $X_2 - S$ in terms of $X_1$ and $X_3$.

$$X_2 - S = \gamma X_1 - S = c \gamma X_3 - S$$

In the first case $X_1 < S$ the two portfolios have the same value, and in the second case $\gamma X_1 < S$ the term $c \gamma X_1 - S$ is negative, which implies that the second portfolio is worth more than the first portfolio. Thus, no matter the value of the stock at expiration, the second portfolio is worth at least as much as the first portfolio. Hence, they are so related when established. That is,

$$P_2 = c P_1 + \gamma X_1 < c P_3$$.

3. (10) Work problems 2, 3, and 4 on page 90 of the text.