1. (10) Use Ito’s lemma to compute the differentials of the following functions:

The thing to remember is that \((dB)^2\) is replaced with \(dt\).

(a) \((t + B_t)^2\)

If \(df (t, B)\) is expanded in a Taylor series we have

\[
df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial B} dB + \frac{1}{2} \frac{\partial^2 f}{\partial B^2} (dt)^2 + \frac{\partial^2 f}{\partial t \partial B} dt dB + \frac{1}{2} \frac{\partial^2 f}{\partial B^2} (dB)^2 + \cdots
\]

\[
\approx \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial B} dB + \frac{1}{2} \frac{\partial^2 f}{\partial B^2} (dt)^2 = \left( \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial B^2} \right) dt + \frac{\partial f}{\partial B} dB
\]

All terms which have factors which tend to zero faster than \(dt\) have been discarded to get the last expression. Thus, we have

\[
df = (2(t + B) + 1) dt + 2(t + B) dB
\]

(b) \(F (S, t) = tS^2 + e^{\sigma S}\), where \(dS = \mu S dt + \sigma S dB\)

Arguing as we did in part a. we have for a function \(f (t, S)\)

\[
df = \left( \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt + \frac{\partial f}{\partial S} dS
\]

\[
= \left( S^2 + \frac{1}{2} \sigma^2 S^2 (2t + \sigma^2 e^{\sigma S}) \right) dt + (2t \sigma e^{\sigma S}) dS.
\]

2. (10) Let \(C(S, \tau)\) denote the value of a european call option, with strike price \(X = 10\), where \(S\) denotes the value of a share of stock and \(\tau\) is the time to expiration. Assume the interest rate is \(e^r = 1.025\), i.e., 2.5 percent, and that the volatility of the stock is 0.25.

(a) Fill in the table below.

<table>
<thead>
<tr>
<th></th>
<th>9.5</th>
<th>9.75</th>
<th>10</th>
<th>10.25</th>
<th>10.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C(S, 1/4))</td>
<td>0.296</td>
<td>0.402</td>
<td>0.528</td>
<td>0.674</td>
<td>0.838</td>
</tr>
<tr>
<td>(C(S, 1/2))</td>
<td>0.51</td>
<td>0.629</td>
<td>0.763</td>
<td>0.911</td>
<td>1.071</td>
</tr>
<tr>
<td>(C(S, 1))</td>
<td>0.835</td>
<td>0.967</td>
<td>1.109</td>
<td>1.261</td>
<td>1.422</td>
</tr>
</tbody>
</table>
(b) For $5 \leq S \leq 15$, graph the function $C(S,0)$, $C(S,1/4)$, $C(S,1/2)$, and $C(S,1)$ in a common plot.

3. (15) Let $C$ denote the price of a European call option. Then $C = C(s,x)$. That is, $C$ depends upon the stock price $s$ and the strike price $x$.

(a) Show that $C(s,x) = xsC \left( \frac{1}{x}, \frac{1}{s} \right)$.

The formulas for $d_1$ and $d_2$ show that $d_1 \left( \frac{1}{x}, \frac{1}{s} \right) = d_1(s,x)$. Thus,

$$C \left( \frac{1}{x}, \frac{1}{s} \right) = \frac{1}{x} \left[ sN(d_1(s,x)) - xe^{-rt}N(d_2(s,x)) \right]$$

$$= \frac{1}{x} C(s,x).$$

(b) Find simple formulas for $\frac{\partial C}{\partial x}$ and $\frac{\partial^2 C}{\partial x^2}$.

$$\frac{\partial}{\partial x} C(s,x) = \frac{s}{x} \left( \frac{1}{x} C \left( \frac{1}{x}, \frac{1}{s} \right) \right)$$

$$= \frac{1}{x} C \left( \frac{1}{x}, \frac{1}{s} \right) - \frac{s}{x} \left( d_1 \left( \frac{1}{x}, \frac{1}{s} \right) - \frac{1}{x} \right)$$

From this formula we see that $C$ is a decreasing function of $x$ and that $-1 < \frac{\partial C}{\partial x} < 0$. The second derivative equals:
\[
\begin{align*}
\frac{\partial^2 C}{\partial x^2} &= \frac{\partial}{\partial x} \left( -e^{-r\tau} N \left( d_2 \left( s, x \right) \right) \right) = -e^{-r\tau} N' \left( d_2 \right) \frac{\partial d_2}{\partial x} \\
&= -\frac{e^{-r\tau}}{\sqrt{2\pi}} \frac{e^{-d_2^2/2}}{\sigma x \sqrt{\tau}} \left( -\frac{1}{\sigma x \sqrt{\tau}} \right) \\
&= \frac{e^{-r\tau}}{\sigma x \sqrt{\tau} \sqrt{2\pi}} e^{-d_2^2/2}
\end{align*}
\]

(c) Based upon the results from part b, what can you deduce about the graph of \( C \) as a function of the strike price \( x \)?

The first derivative is always less than 0, and the second derivative is always positive. Thus, the graph of \( C \) as a function of the strike price is decreasing and concave up.

4. (10) Let \( C \) denote the price of a European call option. Then

\[
C = C \left( s, x, r, \tau, \sigma \right) = sN \left( d_1 \right) - e^{-r\tau} x N \left( d_2 \right).
\]

Show that

\[
\frac{\partial C}{\partial \sigma} = \frac{\sqrt{\tau} s}{\sqrt{2\pi}} e^{-d_2^2/2},
\]

and conclude that the price of a call option is an increasing function of the asset’s volatility.

The formulas \( \frac{\partial d_2}{\partial \sigma} = \frac{\partial d_1}{\partial \sigma} - \sqrt{\tau} \) and \( e^{-d_2^2/2} = e^{-d_1^2/2} s \) are used in the calculations below.

\[
\begin{align*}
\frac{\partial C}{\partial \sigma} &= sN' \left( d_1 \right) \frac{\partial d_1}{\partial \sigma} - e^{-r\tau} x N' \left( d_2 \right) \frac{\partial d_2}{\partial \sigma} \\
&= \frac{se^{-d_1^2/2} \partial d_1}{\sqrt{2\pi}} \frac{1}{\sigma} - e^{-r\tau} x e^{-d_1^2/2} \frac{1}{\sqrt{2\pi}} \left( \frac{\partial d_1}{\partial \sigma} - \sqrt{\tau} \right) \\
&= \frac{se^{-d_1^2/2} \partial d_1}{\sqrt{2\pi}} \frac{1}{\sigma} - \sqrt{\tau} x e^{-d_1^2/2} \frac{1}{\sqrt{2\pi}} \left( \frac{\partial d_1}{\partial \sigma} - \sqrt{\tau} \right) \\
&= \frac{s}{\sqrt{2\pi}} e^{-d_1^2/2} \partial d_1 \frac{1}{\sigma} - \frac{s}{\sqrt{2\pi}} e^{-d_1^2/2} \left( \frac{\partial d_1}{\partial \sigma} - \sqrt{\tau} \right) \\
&= \frac{\sqrt{\tau} s}{\sqrt{2\pi}} e^{-d_2^2/2}
\end{align*}
\]

Thus, the price of a European call option is an increasing function of the stock’s volatility.