1. (20) For the stock you are watching calculate at least three different implied volatilities. Use call options with the same expiration date, but different strike prices. Plot these values to see if you get a "volatility smile".

2. (20) A certain stock has a price volatility of $\sigma = 0.25$, and will pay a one time dividend at a rate of $d_y = 0.01$ in a month’s time. If the current price of the stock is $41$, determine the price of a European call option, which expires in three months and has a strike price of $40$. Assume that $e^r = 1.02$.

For a European call option on a one-time dividend paying stock there is an easy to use formula for the value of this option, and it is

$$C_d(S, t) = (1 - d_y) C\left(S, t, (1 - d_y)^{-1} X\right).$$

Thus, for this particular option we have

$$C_d(41, 0) = (0.99) C\left(41, 0, \frac{40}{0.99}\right) = 2.4215.$$

The Black-Scholes formula was used to evaluate the term $C(41, 0, 40/0.99)$. 

3. A certain stock has price volatility equal to 0.25. Assume that the current annual interest rate is \( r = \ln(1.015) \) %. That is, \( e^r = 1.015 \). A broker decides to issue the following option. Its strike price is $10, and when it matures in 3 months it can be exercised with the following payout

\[
V(S, 1/4) = \begin{cases} 
0 & \text{if } S < 10 \\
1 & \text{if } 10 \leq S < 12 \\
2 & \text{if } 12 \leq S < 14 \\
5 & \text{if } 14 \leq S
\end{cases}
\]

What should the broker charge for this option, if the stock is currently selling for $10.75? Hint, use the conversion of the Black-Scholes partial differential equation to the heat equation.

The equation which relates the function \( V \) to a function \( u \), which solves the heat equation is:

\[
V(S, t) = E e^{\alpha \ln(S/E)} e^{\beta(T-t)} e^{\frac{\sigma^2}{2} u \left( \ln \left( \frac{S}{E} \right), \frac{\sigma^2}{2} (T-t) \right)},
\]

where \( \alpha = \frac{\sigma^2 - 2r}{2\sigma^2} = 0.26178 \), \( \beta = -\left( \frac{\sigma^2 + 2r}{2\sigma^2} \right)^2 = -0.54497 \), and

\[
u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} u(x, 0) e^{-\frac{(x-\xi)^2}{4t}} d\xi.
\]

To calculate \( u(x, t) \) we need to know what \( u(x, 0) \) equals. We observe that in the equation, which relates \( V \) to \( u \), the second slot of \( u \) will equal 0 if \( t = T \). This leads to the equation

\[
V(S, T) = E e^{\alpha \ln(S/E)} u \left( \ln \left( \frac{S}{E} \right), 0 \right).
\]

Setting \( \xi = \ln(S/E) \) or \( S = E e^{\xi} \), and solving the above equation for \( u \) we have

\[
u(\xi, 0) = \frac{1}{E} e^{-\alpha \xi} V \left( E e^{\xi}, T \right),
\]

where

\[
V \left( E e^{\xi}, T \right) = \begin{cases} 
0 & \text{if } \xi < 0 \\
1 & \text{if } 0 \leq \xi < \ln \left( \frac{12}{10} \right) \\
2 & \text{if } \ln \left( \frac{12}{10} \right) \leq \xi < \ln \left( \frac{14}{10} \right) \\
5 & \text{if } \ln \left( \frac{14}{10} \right) \leq S
\end{cases}
\]

Thus,

\[
u(\xi, 0) = \frac{1}{E} e^{-\alpha \xi} \begin{cases} 
0 & \text{if } \xi < 0 \\
1 & \text{if } 0 \leq \xi < \ln \left( \frac{12}{10} \right) \\
2 & \text{if } \ln \left( \frac{12}{10} \right) \leq \xi < \ln \left( \frac{14}{10} \right) \\
5 & \text{if } \ln \left( \frac{14}{10} \right) \leq S
\end{cases}
\]
Since the stock price is 10.75 and $T = 1/4$, we want the value of

$$
V (10.75, 0) = 10e^{\alpha \ln(1.075)}e^{\beta \sigma^2 / 8}u \left( \ln \left( \frac{S}{E} \right), \frac{\sigma^2}{2}T \right)
$$

$$
= 10.1478 u \left( \ln \left( \frac{S}{E} \right), \frac{\sigma^2}{2}T \right)
$$

$$
= 0.932
$$