Chapter 2. First-Order Differential Equations

Example 1. Solve the equation

\[ x^2 y^2 y' + 1 = y. \]

SOLUTION Separating the variables and integrating gives

\[ \frac{y^2}{y - 1} dy = \frac{dx}{x}, \]

\[ \int \frac{y^2}{y - 1} dy = \int \frac{dx}{x}. \]

So, the implicit solution to the equation is

\[ \frac{y^2}{2} + y + \ln |y - 1| = -\frac{1}{x} + C. \]

Section 2.3 Linear Equations

A linear first-order equation is an equation that can be expressed in the form

\[ a_1(x) \frac{dy}{dx} + a_0(x)y = b(x), \tag{1} \]

where \( a_0(x), a_1(x), b(x) \) depend only on \( x \).

We will assume that \( a_0(x), a_1(x), b(x) \) are continuous functions of \( x \) on an interval \( I \).

For now, we are interested in those linear equations for which \( a_1(x) \) is never zero on \( I \). In that case we can rewrite (1) in the standard form

\[ \frac{dy}{dx} + P(x)y = Q(x), \tag{2} \]

where \( P(x) = a_0(x)/a_1(x) \) and \( Q(x) = b(x)/a_1(x) \) are continuous on \( I \).

There are two methods of solving linear first-order differential equations.

Method 1.

Equation (2) can be solved by finding an integrating factor \( \mu(x) \) such that

\[ \frac{d}{dx} [\mu y] = \frac{dy}{dx} \mu + \frac{d\mu}{dx} y = \mu Q(x). \]

Since,

\[ \frac{d}{dx} [\mu y] = Q(x) - P(x)y, \]

\[ \frac{d}{dx} [\mu y] = \frac{dy}{dx} \mu + \frac{d\mu}{dx} y = (Q(x) - P(x)y)\mu + \frac{d\mu}{dx} y - P(x)y\mu + \mu Q(x) = \left( \frac{d\mu}{dx} - P(x)\mu \right) y + \mu Q(x) = \mu Q(x). \]
Clearly, this requires that \( \mu \) satisfy
\[
\frac{d\mu}{dx} - P(x)\mu = 0. \tag{3}
\]
Lets find the solution to equation (3).

\[
\frac{d\mu}{dx} = P(x)\mu,
\]
\[
\frac{d\mu}{\mu} = P(x)dx,
\]
\[
\int \frac{d\mu}{\mu} = \int P(x)dx,
\]
\[
\ln \mu = \int P(x)dx,
\]
\[
\mu = \exp \left[ \int P(x)dx \right]. \tag{4}
\]

With this choice for \( \mu(x) \), equation (2) becomes
\[
\frac{d}{dx} [\mu(x)y] = \mu(x)Q(x),
\]
\[
\mu(x)y = \int \mu(x)Q(x)dx + C,
\]
\[
y = \frac{1}{\mu} \int \mu(x)Q(x)dx + C,
\]
here \( C \) is an arbitrary constant.
The solution (5) to equation (2) is called the \textbf{general solution}.

\textbf{Example 2.} Obtain the general solution to the equation
\[
xy' - y = -\ln x.
\]

\textbf{SOLUTION} \quad \text{To put this equation in standard form, we divide by } x \text{ to obtain}
\[
y' - \frac{y}{x} = -\frac{\ln x}{x}.
\]
Here \( P(x) = -\frac{1}{x}, \) \( Q(x) = -\frac{\ln x}{x} \).
We can find the integrating factor \( \mu(x) \) solving the equation
\[
\frac{d\mu}{dx} + \frac{1}{x}\mu = 0.
\]
Separating the variables and integrating gives
\[
\frac{d\mu}{\mu} = -\frac{dx}{x},
\]
\[
\int \frac{d\mu}{\mu} = -\int \frac{dx}{x},
\]
\[
\ln \mu = -\ln x,
\]
\[
\mu(x) = \frac{1}{x}.
\]
Now, we can find \( y \) from the equation
\[
\frac{d}{dx} \left[ -\frac{1}{x} y \right] = \frac{1}{x} \ln x = \frac{\ln x}{x^2}
\]
Integrating gives
\[
-\frac{y}{x} = \int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} - \frac{1}{x} + C.
\]
Thus, the general solution to the given equation is
\[
y(x) = \ln x + 1 + Cx.
\]

**Method 2 (variation of parameter).**

Associated with equation \( y' + P(x)y = Q(x) \) is the equation
\[
y' + P(x)y = 0, \tag{6}
\]
which is obtained from (2) by replacing \( Q(x) \) with zero. We say that equation (2) is a nonhomogeneous equation and that (6) is the corresponding homogeneous equation.

We can obtain the general solution to the nonhomogeneous equation solving homogeneous equation
\[
y' + P(x)y = 0.
\]
The solution to homogeneous equation is
\[
y_{\text{hom}}(x) = C \exp \left[ -\int P(x) dx \right],
\]
then the general solution to the nonhomogeneous equation is
\[
y(x) = C(x) \exp \left[ -\int P(x) dx \right], \tag{7}
\]
where \( C(x) \) is an unknown function that depends on \( x \).
Since
\[ y'(x) = C'(x) \exp \left[ - \int P(x) \, dx \right] - C(x) P(x) \exp \left[ - \int P(x) \, dx \right], \quad (8) \]

substitution of right parts of (8) and (7) for \( y \) and \( y' \) in nonhomogeneous equation gives

\[
C'(x) \exp \left[ - \int P(x) \, dx \right] - C(x) P(x) \exp \left[ - \int P(x) \, dx \right] + P(x) C(x) \exp \left[ - \int P(x) \, dx \right] = Q(x),
\]

\[
C'(x) \exp \left[ - \int P(x) \, dx \right] = Q(x),
\]

\[
C'(x) = Q(x) \exp \left[ \int P(x) \, dx \right],
\]

\[
C'(x) = \int Q(x) \exp \left[ \int P(x) \, dx \right] \, dx + C_1,
\]

where \( C_1 \) is an arbitrary constant. Thus, the general solution to nonhomogeneous equation is

\[
y(x) = \exp \left[ - \int P(x) \, dx \right] \int Q(x) \exp \left[ \int P(x) \, dx \right] \, dx + C_1.
\]

**Example 3.** Obtain the general solution to the equation

\[
xy' - y = - \ln x
\]

using Method 2.

**SOLUTION.** The corresponding homogeneous equation to the given equation is

\[
xy' - y = 0.
\]

Separating the variables and integrating gives

\[
\frac{dy}{y} = \frac{dx}{x},
\]

\[
\int \frac{dy}{y} = \int \frac{dx}{x},
\]

\[
y_{\text{hom}} = Cx.
\]

The general solution to the given nonhomogeneous equation is

\[
y(x) = C(x) x,
\]

where \( C(x) \) is an unknown function.

\[
y'(x) = C'(x) x + C(x).
\]

Then
\[ C'(x)x^2 + C(x)x - C(x)x = -\ln x, \]

\[ C'(x) = -\frac{\ln x}{x^2}, \]

\[ C(x) = -\int \frac{\ln x}{x^2} dx = \frac{\ln x}{x} + \frac{1}{x} + C_1. \]

Thus, the general solution to the given nonhomogeneous equation is

\[ y(x) = \left( \frac{\ln x}{x} + \frac{1}{x} + C_1 \right) x = \ln x + 1 + C_1 x. \]

**Existence and uniqueness of solution**

**Theorem 1.** Suppose \( P(x) \) and \( Q(x) \) are continuous on some interval \( I \) that contains the point \( x_0 \). Then for any choice of initial value \( y_0 \), there exists a unique solution \( y(x) \) on \( I \) to the initial value problem

\[ y' + P(x)y = Q(x), \quad y(x_0) = y_0. \quad (9) \]

In fact, the solution is given by (5) for a suitable value of \( C \).