Problem 1. Using the definition of weak derivative, find the values of $\alpha$ for which $x^\alpha$ has a weak derivative in $L^2(0,1)$?

Problem 2. Let $\Omega = \{(x, y) \in \mathbb{R}^2, |(x, y)| < 1/2\}$ and consider
$$u(x, y) = \ln(\ln(1/r)), \quad \text{for } r \neq 0.$$ Here $r = \sqrt{x^2 + y^2}$.

(a) Write the integral
$$\int_{\Omega} u(x, y)^2 \, da$$
in terms of polar integration and show that the resulting integral is finite, i.e., $u$ is in $L^2(\Omega)$.

(b) For $r \neq 0$, calculate
$$\nabla u = \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right)$$
in terms of $u_r, u_\theta, r$ and $\theta$.

(c) Show that the weak derivative of $\nabla u$ exists and is given by the "strong" derivative just computed.

(d) Use integration in polar coordinates to evaluate (or bound)
$$\int_{\Omega} |\nabla u|^2 \, da.$$ As this integral is finite, $u$ is in $H^1(\Omega)$.

The function just constructed gives an example of a function in $H^1(\Omega)$ with $\Omega \subset \mathbb{R}^2$ which goes to infinity at a point. Obviously, this function is not continuous. When $\Omega$ is in $\mathbb{R}^3$, $H^1(\Omega)$ contains functions which blow up on lines.

Problem 3. Let $\Omega = (0, 2) \times (0, 1)$, $\Omega_1 = (0, 1)^2$, and $\Omega_2 = (1, 2) \times (0, 1)$. Suppose that $f$ is in $C^\infty(\Omega_i)$, for $i = 1, 2$ (the set of infinitely differentiable functions which, along with their derivatives, are continuous on $\Omega_i$). Show that $\partial f/\partial x$ exists in $L^1_{\text{loc}}(\Omega)$ if and only if $f$ is continuous across $\Gamma = \overline{\Omega_1} \cap \overline{\Omega_2}$. When $f$ is continuous across $\Gamma$, $\partial f/\partial x(x, y) = f_x(x, y)$ for all $(x, y) \in \Omega_1 \cup \Omega_2$ but may be undefined for $(x, y) \in \Gamma$.

For the last problem, you may assume:
(a) A smooth function $g$ on $\Gamma$ is identically zero if and only if
\[ \int_\gamma g\phi = 0, \text{ for all } \phi \in C_0^\infty(\Gamma). \]

(b) There is a function $\chi$ in $C^\infty(0,2)$ which is supported on $[1/2,3/2]$ and satisfies $\chi(1) = 1.$