Homework #3. (Feb. 5)
Math. 610:600
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Problem 1. Suppose that \( m \in C^1[0, 1] \), \( v \in C^\infty[0, 1] \) and \( \phi \in C^\infty_0(0, 1) \). By integration by parts, we clearly have

\[-(mv, \phi') = (m'v + mv', \phi).\]

Use this and the density of \( C^\infty[0, 1] \) in \( H^1(0, 1) \) to show that the weak derivative of \( mw \) exists for \( w \in H^1(0, 1) \) and that \( mw \) is in \( H^1(0, 1) \).

Problem 2. Consider the problem:

\[
\begin{align*}
u - (au)' &= f \quad \text{in} \ (0, 1), \\
u'(0) &= 0, \quad u'(1) + \alpha u(1) = 0.
\end{align*}
\]

(The boundary condition at one is called a “Robin” boundary condition.) Initially, we assume that \( a \in L^\infty(0, 1) \) and satisfies \( 0 < a_0 \leq a(x) \leq a_1 \) a.e. for \( x \in (0, 1) \).

(a) Derive a variational formulation of the above problem on \( H^1(0, 1) \). (Hint: All terms involving both \( u \) and the test function \( \phi \) are part of the bilinear form.)

(b) Show that the resulting bilinear form is bounded.

(c) Using the trace theorem, show that there is a constant \( c_0 > 0 \) such that if \( \alpha > -c_0 \), then the resulting bilinear form is coercive. In this case, there is a unique solution (in \( H^1(0, 1) \)) to the variational problem provided that \( f \in L^2(0, 1) \).

(d) Let \( u \) be the unique solution in \( H^1(0, 1) \) satisfying the variational equations. Show that \( w = au' \) is in \( H^1(0, 1) \). Hint: use the variational equations to show that the weak derivative of \( w \) exists and is in \( L^2(0, 1) \).

(e) Use Problem 1 to show that \( u \in H^2(0, 1) \) when \( a(x) \in C^1[a, b] \) (and \( f \in L^2(0, 1) \)).