Multiple Choice: (4 points each)

1. Compute \( \int_0^{\pi/2} x \cos(3x) \, dx \)
   a. \( -\frac{\pi}{6} - \frac{1}{3} \)
   b. \( -\frac{\pi}{3} - \frac{1}{3} \)
   c. \( -\frac{\pi}{3} + \frac{1}{3} \)
   d. \( -\frac{\pi}{6} - \frac{1}{9} \)  correct choice
   e. \( -\frac{\pi}{3} + \frac{1}{9} \)

Integrate by parts with \( u = x \)  \( dv = \cos(3x) \, dx \)
   \( du = dx \)  \( v = \frac{1}{3} \sin(3x) \)

\[
\int_0^{\pi/2} x \cos(3x) \, dx = \left[ \frac{x}{3} \sin(3x) - \frac{1}{3} \int \sin(3x) \, dx \right]_0^{\pi/2} = \left[ \frac{x}{3} \sin(3x) + \frac{1}{9} \cos(3x) \right]_0^{\pi/2}
\]

\[
= \frac{\pi}{6} \sin\left(\frac{3\pi}{2}\right) - \frac{1}{9} \cos(0) = -\frac{\pi}{6} - \frac{1}{9}
\]

2. Compute \( \lim_{n \to \infty} \frac{2^n}{1 + 3^n} \)
   a. 0  correct choice
   b. \( \frac{1}{2} \)
   c. \( \frac{1}{1 - \frac{2}{3}} \)
   d. \( \frac{1}{2} \)
   e. \( \frac{1}{1 - \frac{2}{3}} \)

\[
\lim_{n \to \infty} \frac{2^n}{1 + 3^n} = \lim_{n \to \infty} \frac{2^n}{\frac{3^n}{3^n} + 1} = \frac{0}{1} = 0
\]

3. Compute \( \int_0^{\pi/2} \sin^3 \theta \, d\theta \)
   a. \( -\frac{2}{3} \)
   b. \( -\frac{1}{3} \)
   c. 0
   d. \( \frac{1}{3} \)
   e. \( \frac{2}{3} \)  correct choice

Let \( u = \cos \theta \). Then \( du = -\sin \theta \, d\theta \). So:

\[
\int_0^{\pi/2} \sin^3 \theta \, d\theta = \int_0^{\pi/2} (1 - \cos^2 \theta) \sin \theta \, d\theta = -\int_1^0 (1 - u^2) \, du
\]

\[
= -\left[ u - \frac{u^3}{3} \right]_1^0 = -0 + \left[ 1 - \frac{1}{3} \right] = \frac{2}{3}
\]
4. Which formula will give the arclength of the curve \( y = \sin x \) between \( x = 0 \) and \( x = \pi \)?

   a. \( L = \int_{0}^{\pi} 2\pi x \sqrt{1 + \cos^2 x} \, dx \)
   b. \( L = \int_{0}^{\pi} \sqrt{1 + \cos^2 x} \, dx \) \( \text{correct choice} \)
   c. \( L = \int_{0}^{\pi} 2\pi \sin x \sqrt{1 + \cos^2 x} \, dx \)
   d. \( L = \int_{0}^{\pi} 2\pi x \sqrt{1 + \sin^2 x} \, dx \)
   e. \( L = \int_{0}^{\pi} \sqrt{1 + \sin^2 x} \, dx \)

\[
\frac{dy}{dx} = \cos x \quad L = \int_{0}^{\pi} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_{0}^{\pi} \sqrt{1 + \cos^2 x} \, dx
\]

5. Which initial value problem describes the solution to the following problem:
A 100 gal tank is initially filled with sugar water whose concentration is \( 0.05 \) lb sugar gal water. Sugar is added to the tank at the rate of \( 2 \text{ lb} \text{ hr}^{-1} \) and pure water is added at the rate of \( 3 \text{ gal} \text{ hr}^{-1} \). The mixture is kept well mixed and drained at the rate of \( 3 \text{ gal} \text{ hr}^{-1} \). Find the amount of sugar in the tank after \( t \) hours.

   a. \( \frac{dS}{dt} = 2 - 0.03S, \quad S(0) = 5 \) \( \text{correct choice} \)
   b. \( \frac{dS}{dt} = 0.1 - 0.15S, \quad S(0) = 5 \)
   c. \( \frac{dS}{dt} = 3S - 0.02, \quad S(0) = 0.05 \)
   d. \( \frac{dS}{dt} = 0.02 - 3S, \quad S(0) = 5 \)
   e. \( \frac{dS}{dt} = 0.02 - 0.03S, \quad S(0) = 0.05 \)

\[
\frac{dS}{dt} \text{ lb} \text{ hr}^{-1} = \frac{2 \text{ lb} \text{ hr}^{-1} - \frac{S}{100 \text{ gal}} \cdot 3 \text{ gal} \text{ hr}^{-1}}{100 \text{ gal}} = 2 - 0.03S \quad S(0) = 0.05 \text{ lb gal}^{-1} 100 \text{ gal} = 5
\]

6. Find the solution of the differential equation \( \frac{dy}{dx} = 2x(1 + y^2) \) satisfying the initial condition \( y(2) = 0 \).

   a. \( y = \tan(x^2) + 2 \)
   b. \( y = \tan^2(x - 2) \)
   c. \( y = \tan(x^2 - 4) \) \( \text{correct choice} \)
   d. \( y = \tan(x^2 + \arctan 2) \)
   e. \( y = \tan^2(x) - \tan^2 2 \)

\[
\int \frac{dy}{1+y^2} = \int 2x \, dx \quad \arctan y = x^2 + C \quad \arctan 0 = 4 + C \quad y = \tan(x^2 - 4)
\]
7. Compute \( \int_{1}^{2} \frac{1}{(x - 2)^{2/3}} \, dx \)

a. \(-\infty\)

b. \(-3\)

c. \(-1\)

d. \(3\) correctchoice

e. \(\infty\)

\( \int_{1}^{2} \frac{1}{(x - 2)^{2/3}} \, dx = \left[ \frac{3(x - 2)^{1/3}}{1} \right]_{1}^{2} = 3(2 - 2)^{1/3} - 3(1 - 2)^{1/3} = 3 \)

8. Compute \( \lim_{x \to 0} \frac{\sin(2x) - 2x}{3x^3} \)

a. \(\frac{1}{9}\)

b. \(-4\)

c. \(-\frac{4}{9}\) correctchoice

d. \(-\frac{8}{9}\)

e. \(-\frac{4}{3}\)

\( \lim_{x \to 0} \frac{\sin(2x) - 2x}{3x^3} = \lim_{x \to 0} \left[ \frac{2x - \frac{(2x)^3}{3!} + \cdots}{3x^3} \right] - \frac{2x}{\lim_{x \to 0} \left[ \frac{2x - \frac{(2x)^3}{3!} + \cdots}{3x^3} \right]} = \frac{2}{6} = -\frac{4}{9} \)

9. Find the radius of convergence of the series \( \sum_{n=1}^{\infty} \frac{2^n}{(n+1)^2} (x - 3)^n \).

a. \(0\)

b. \(\frac{1}{2}\) correctchoice

c. \(2\)

d. \(\frac{1}{3}\)

e. \(3\)

\( L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{2^{n+1}(x - 3)^{n+1}}{(n + 2)^2} \frac{(n + 1)^2}{2^n (x - 3)^n} \right| = 2|x - 3| \)

Convergent if \( L = 2|x - 3| < 1 \) or \(|x - 3| < \frac{1}{2}\). So \( R = \frac{1}{2} \).
10. Which term is incorrect in the following partial fraction expansion?

\[
\frac{-10x^2 + 5x^3 - 8x + 1}{(x - 1)(x - 3)^2(x^2 + 2)} = \frac{A}{x - 1} + \frac{B}{x - 3} + \frac{D}{(x - 3)^2} + \frac{Ex + F}{x^2 + 2}
\]

a. 

b. 

c. 

d. 

e. They are all correct. 

correctchoice

A linear or linear to a power in the denominator gets a constant in the numerator.
A quadratic or quadratic to a power in the denominator gets a linear in the numerator.
So all terms are correct.

11. A vector \( \vec{u} \) has length 3. A vector \( \vec{v} \) has length 4. The angle between them is 60°. Find \( \vec{u} \cdot \vec{v} \).

a. 6  
correctchoice

b. \( \frac{1}{24} \)

c. \( \frac{\sqrt{3}}{24} \)

d. 24

e. \( 6\sqrt{3} \)

\[
\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos \theta = 3 \cdot 4 \cdot \cos 60° = \frac{12}{2} = 6
\]

12. Find an equation for the plane containing the two lines

\[ L_1 : \quad x = 3 + 3t \quad y = 1 + 4t \quad z = 2 + 5t \]
\[ L_2 : \quad x = 3 + t \quad y = 1 \quad z = 2 - t \]

a. \( -4x - 8y - 4z = 10 \)

d. \( x + 2y + z = 7 \)

e. \( x + 2y + z = 10 \)

correctchoice

\[ \vec{v}_1 = (3, 4, 5) \quad \vec{v}_2 = (1, 0, -1) \quad \vec{N} = \vec{v}_1 \times \vec{v}_2 = (-4, 8, -4) \quad P = (3, 1, 2) \]

\[
\vec{N} \cdot (X - P) = 0 \quad -4(x - 3) + 8(y - 1) - 4(z - 2) = 0
\]
\[
-4x + 8y - 4z = -12 \quad x - 2y + z = 3
\]
Work Out (13 points each)

Show all your work. Partial credit will be given. You may not use a calculator.

13. Compute \[ \int \frac{\sqrt{x^2 - 1}}{x} \, dx \]

   \( x = \sec \theta \quad dx = \sec \theta \tan \theta \, d\theta \quad \sqrt{x^2 - 1} = \sqrt{\sec^2 \theta - 1} = \tan \theta \)

   \[ \int \frac{\sqrt{x^2 - 1}}{x} \, dx = \int \frac{\tan \theta}{\sec \theta} \sec \theta \tan \theta \, d\theta \]

   \[ = \int \tan^2 \theta \, d\theta \]

   \[ = \int 1 - \sec^2 \theta \, d\theta \]

   \[ = \theta - \tan \theta \]

   \[ = \arcsin x - \sqrt{x^2 - 1} + C \]

14. The parametric curve given by \( x = t^2, \quad y = \frac{2}{3} t^3, \quad z = \frac{1}{4} t^4 \) for \( 0 \leq t \leq 2 \) is rotated about the \( y \)-axis. Find the area of the surface of revolution.

HINT: Factor the quantity in the square root.

\[ \frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 2t^2, \quad \frac{dz}{dt} = t^3 \]

\[ A = \int_0^2 2\pi x \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2} \, dt \]

\[ = \int_0^2 2\pi t^2 \sqrt{(2t)^2 + (2t^2)^2 + (t^3)^2} \, dt \]

\[ = \int_0^2 2\pi t^2 \sqrt{4t^2 + 4t^4 + t^6} \, dt \]

\[ = \int_0^2 2\pi t^2 \sqrt{t^2 (2 + t^2)^2} \, dt = \int_0^2 2\pi t^3 \sqrt{(2 + t^2)^2} \, dt = 2\pi \int_0^2 (2t^2 + t^4) \, dt \]

\[ = 2\pi \left[ \frac{t^4}{2} + \frac{t^6}{6} \right]_0 \]

\[ = 2\pi \left[ \frac{16}{2} + \frac{64}{6} \right] = 16\pi \left( 1 + \frac{4}{3} \right) = \frac{112}{3} \pi \]
15. The region in the first quadrant between the curves \( y = x^2 \) and \( y = 6 - x \) is rotated about the \( y \)-axis. Find the volume of the solid of revolution.

Use an \( x \)-integral with cylinders. \( h = 6 - x - x^2 \) \( r = x \)

To find the right endpoint, we solve \( x^2 = 6 - x \), or \( x^2 + x - 6 = 0 \)
or \( (x - 2)(x + 3) = 0 \). In the first quadrant \( x = 2 \).

\[
V = \int_0^2 2\pi rh \, dx = \int_0^2 2\pi x(6 - x - x^2) \, dx = 2\pi \int_0^2 (6x - x^2 - x^3) \, dx \\
= 2\pi \left[ 3x^2 - \frac{x^3}{3} - \frac{x^4}{4} \right]_0^2 = 2\pi \left[ 12 - \frac{8}{3} - 4 \right] = \frac{32}{3} \pi
\]

16. A water tank has the shape of a circular cylinder laying on its side. It is 3 ft in radius and 5 ft long. It is half full of water. How much work is needed to pump the water out a spout at the top? (The weight density of water is \( \rho g = 64.5 \text{ lb/ft}^3 \) but you may leave your answer as a multiple of \( \rho g \).)

We put the origin at the center of a circular end with \( y \) measured downward. So the water at height \( y \) must be lifted a distance \( D = y + 3 \).

To know the weight of a slab of water at height \( y \), we must know its volume. Its length is 5. Its width is \( 2x \). Its thickness is \( dy \). By the Pythagorean theorem \( x = \sqrt{9 - y^2} \). So the weight is

\[
dF = \rho g \, dV = \rho g 10x \, dy = \rho g 10\sqrt{9 - y^2} \, dy.
\]

So the work is

\[
W = \int D \, dF = \int_0^3 (y + 3) \rho g 10\sqrt{9 - y^2} \, dy \\
= 10\rho g \int_0^3 y\sqrt{9 - y^2} \, dy + 30\rho g \int_0^3 \sqrt{9 - y^2} \, dy
\]

The first integral is a simple substitution. The second integral is the area of a quarter circle of radius 3.

\[
W = 10\rho g \left[ -\frac{1}{3} (9 - y^2)^{3/2} \right]_0^3 + 30\rho g \frac{1}{4} \pi (3)^2 \\
= 10\rho g \frac{1}{3} (9)^{3/2} + \frac{135}{2} \rho g \pi = \rho g \left( 90 + \frac{135}{2} \pi \right) \\
= 64.5 \left( 90 + \frac{135}{2} \pi \right)
\]