Multiple Choice: (4 points each)

1. (4 points) If \( \vec{F} = (2yz, -2xz, x^2z + y^2z) \), compute \( \vec{\nabla} \cdot \vec{F} \).
   
   a. \( 2yz - 2xz + 2x + 2y - 4z \)
   b. \( x^2 + y^2 \) Correct Choice
   c. \( (2yz + 2x, 2xz - 2y, -4z) \)
   d. \( (0, 0, x^2 + y^2) \)
   e. \( (2yz + 2x, 2y - 2xz, -4z) \)

   \[
   \vec{\nabla} \cdot \vec{F} = \partial_x(2yz) + \partial_y(-2xz) + \partial_z(x^2z + y^2z) = x^2 + y^2
   \]

2. (4 points) If \( \vec{F} = (2yz, -2xz, x^2z + y^2z) \), compute \( \vec{\nabla} \times \vec{F} \).
   
   a. \( 2yz - 2xz + 2x + 2y - 4z \)
   b. \( x^2 + y^2 \)
   c. \( (2yz + 2x, 2xz - 2y, -4z) \)
   d. \( (0, 0, x^2 + y^2) \)
   e. \( (2yz + 2x, 2y - 2xz, -4z) \) Correct Choice

   \[
   \vec{\nabla} \times \vec{F} = \begin{vmatrix}
   \hat{i} & \hat{j} & \hat{k} \\
   \partial_x & \partial_y & \partial_z \\
   2yz & -2xz & x^2z + y^2z
   \end{vmatrix}
   = i(2yz + 2x) - j(2xz - 2y) + k(-2z - 2z)
   = (2yz + 2x, 2y - 2xz, -4z)
   \]

3. (4 points) If \( \vec{F} = (2yz, -2xz, x^2z + y^2z) \), compute \( \vec{\nabla} \cdot \vec{\nabla} \times \vec{F} \).
   
   a. \( 2x - 2y \)
   b. \( 2x + 2y \)
   c. \( (2, 2, -4) \)
   d. 0 Correct Choice
   e. undefined

   \[
   \vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = 0 \quad \text{for any} \quad \vec{F}.
   \]

   In particular, \( \vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = \partial_x(2yz + 2x) + \partial_y(2y - 2xz) + \partial_z(-4z) = 0 \)
4. (8 points) Find the mass and center of mass of a wire in the shape of the semicircle $x^2 + y^2 = 4$ with $y \geq 0$ if the density is $\rho(x,y) = y$.

Note: By symmetry $\bar{x} = 0$. So you just need to compute $M$ and $\bar{y}$.

$$\vec{r}(\theta) = (2\cos \theta, 2\sin \theta) \quad \vec{v} = (-2\sin \theta, 2\cos \theta) \quad |\vec{v}| = \sqrt{4\sin^2 \theta + 4\cos^2 \theta} = 2 \quad \rho = y = 2\sin \theta$$

$$M = \int \rho\,ds = \int y|\vec{v}|\,d\theta = \int_0^\pi 2\sin \theta 2\,d\theta = [2\cos \theta]_0^\pi = 4 - 4 = 8$$

$$M_x = \int y\rho\,ds = \int y^2|\vec{v}|\,d\theta = \int_0^\pi 4\sin^2 \theta 2\,d\theta = 8 \int_0^\pi \frac{1 - \cos 2\theta}{2}\,d\theta = 4\left[\theta - \frac{\sin 2\theta}{2}\right]_0^\pi = 4\pi$$

$$\bar{y} = \frac{M_x}{M} = \frac{4\pi}{8} = \frac{\pi}{2}$$

5. (8 points) Compute $\iint \vec{V} \times \vec{F}\,dS$ over the cone $z = \sqrt{x^2 + y^2}$ for $z \leq 4$ with normal pointing down and out, for the vector field $\vec{F} = (2yz, -2xz, x^2z + y^2z)$.

Note: The cone may be parametrized by $\vec{R}(r, \theta) = (r\cos \theta, r\sin \theta, r)$. Follow these steps:

$$\vec{r}_r = \left( \cos \theta, \sin \theta, 1 \right) \quad \vec{N} = \hat{i}(-r\cos \theta) - \hat{j}(r\sin \theta) + \hat{k}(r\cos^2 \theta + r\sin^2 \theta) = (-r\cos \theta, -r\sin \theta, r)$$

$$\vec{r}_\theta = (-r\sin \theta, r\cos \theta, 0) \quad \text{which points up and in.}$$

Rev: $\vec{N} = (r\cos \theta, r\sin \theta, -r)$

$$\vec{V} \times \vec{F} = \left| \begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 2yz & -2xz & x^2z + y^2z \end{array} \right| = \hat{i}(2yz + 2x) - \hat{j}(2xz - 2y) + \hat{k}(-2z - 2z)$$

$$= (2yz + 2x, 2y - 2xz, -4z)$$

$$\vec{V} \times \vec{F}(\vec{R}(r, \theta)) = (2r^2 \sin \theta + 2r\cos \theta, 2r\sin \theta - 2r^2 \cos \theta, -4r)$$

$$\vec{V} \times \vec{F} \cdot \vec{N} = (2r^2 \sin \theta + 2r\cos \theta)(r\cos \theta) + (2r\sin \theta - 2r^2 \cos \theta)(r\sin \theta) + (-4r)(-r)$$

$$= 2r^2 \cos^2 \theta + 2r^2 \cos^2 \theta + 4r^2 = 6r^2$$

$$\iint \vec{V} \times \vec{F}\,dS = \int_0^{2\pi} \int_0^4 6r^2 \,dr\,d\theta = \int_0^2 \left[ 2r^3 \right]_r^4 \,d\theta = \int_0^{2\pi} 128 \,d\theta = 256\pi$$