1. Find the equation of the line perpendicular to the graph of 
\[ xyz - x^2 - y^2 - z^2 = -8 \]
at the point \((1, 2, 3)\).

Where does this line intersect the \(xz\)-plane?

- a. \((-7, 0, -5)\)
- b. \((-7, 0, 11)\)
- c. \((9, 0, -5)\) Correct Choice
- d. \((9, 0, 11)\)
- e. \((5, 0, -1)\)

Let \( F = xyz - x^2 - y^2 - z^2 \). Then \( \nabla F = (yz - 2x, xz - 2y, xy - 2z) \).

Then the normal at \( P = (1, 2, 3) \) is \( \mathbf{N} = \nabla F \bigg|_P = (4, -1, -4) \).

This is also the tangent vector to the perpendicular line. So \( \mathbf{v} = (4, -1, -4) \).

So the perpendicular line is 
\[ x = 1 + 4t \quad y = 2 - t \quad z = 3 - 4t \]

This intersects the \(xz\)-plane when \( y = 0 \) or \( t = 2 \).
So \( x = 1 + 4t = 9 \) and \( z = 3 - 4t = -5 \).

2. The point \((x, y) = (1, 2)\) is a critical point of the function \( f(x, y) = x^2 + y^2 - 4 \)^2 - 4x - 8y.

Use the Second Derivative Test to classify it as a

- a. local maximum
- b. local minimum Correct Choice
- c. inflection point
- d. saddle point
- e. Test Fails

\[ f_x = 2(x^2 + y^2 - 4)2x - 4 \quad f_y = 2(x^2 + y^2 - 4)2y - 8 \quad f_x(1, 2) = f_y(1, 2) = 0 \]
\[ f_{xx} = 2(2x)2x + 2(x^2 + y^2 - 4)2 \quad f_{yy} = 2(2y)2y + 2(x^2 + y^2 - 4)2 \quad f_{xy} = 2(2y)2x \]
\[ f_{xx}(1, 2) = 8 + 4 = 12 > 0 \quad f_{yy}(1, 2) = 32 + 4 = 36 > 0 \quad f_{xy}(1, 2) = 16 \]
\[ D = f_{xx}f_{yy} - f_{xy}^2 = 12 \cdot 36 - 16^2 = 176 > 0 \quad \text{local minimum} \]
3. Find the center of mass of the triangle with vertices (0,0), (1,1) and (−1,1) if the mass density is \( \rho = y \).

a. \((0, \frac{1}{3})\)  

b. \((0, \frac{1}{2})\)  

c. \((0, \frac{2}{3})\)  

d. \((0, \frac{3}{4})\)  Correct Choice  

e. \((0, \frac{4}{5})\)  

\(\bar{x} = 0\) by symmetry, Use a \(y\)-integral.  
\(0 \leq y \leq 1\)  
\(x = -y\)  
\(x = y\)  
\[
M = \iint \rho \, dA = \int_{0}^{1} \int_{-y}^{y} y \, dx \, dy = \int_{0}^{1} [xy]_{x=-y}^{y} \, dy = 2 \int_{0}^{1} y^2 \, dy = 2 \left[ \frac{y^3}{3} \right]_{y=0}^{1} = \frac{2}{3}
\]
\(y\)-mom = \(M_x = \iint y \rho \, dA = \int_{0}^{1} \int_{-y}^{y} y^2 \, dx \, dy = \int_{0}^{1} [y^2 x]_{x=-y}^{y} \, dy = 2 \int_{0}^{1} y^3 \, dy = 2 \left[ \frac{y^4}{4} \right]_{y=0}^{1} = \frac{1}{2}
\]
\(\bar{y} = \frac{y\text{-mom}}{M} = \frac{M_x}{M} = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}
\]

4. Compute \(\iint (x^2 + y^2) \, dA\) over the region bounded by the polar curve \(r = \theta\) and the \(x\)-axis.

a. \(\frac{\pi^5}{20}\)  Correct Choice  

b. \(\frac{\pi^4}{16}\)  

c. \(\frac{\pi^4}{12}\)  

d. \(\frac{\pi^3}{9}\)  

e. \(\frac{\pi^3}{6}\)  

\[
\int_{0}^{\pi} \int_{0}^{\theta} r^2 \, r \, dr \, d\theta = \int_{0}^{\pi} \left[ \frac{r^4}{4} \right]_{0}^{\theta} \, d\theta = \frac{1}{4} \int_{0}^{\pi} \theta^4 \, d\theta = \frac{1}{4} \left[ \frac{\theta^5}{5} \right]_{0}^{\pi} = \frac{\pi^5}{20}
\]
5. Find the mass of the 1/8 of the solid sphere $x^2 + y^2 + z^2 \leq 16$ in the first octant if the mass density is $\delta = z$.
   
   a. $64\pi$
   b. $16\pi$ Correct Choice
   c. $8\pi$
   d. $4\pi$
   e. $\pi$

   $$M = \iiint \delta \, dV = \iiint z \, dV = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^4 \rho (\cos \varphi) \rho^2 (\sin \varphi) \, d\rho \, d\varphi \, d\theta$$

   $$= \int_0^{\pi/2} d\theta \int_0^{\pi/2} \cos \varphi \sin \varphi \, d\varphi \int_0^4 \rho^3 \, d\rho = \frac{\pi}{2} \left[ \frac{\sin^2 \varphi}{2} \right]_{\varphi=0}^{\pi/2} \left[ \frac{\rho^4}{4} \right]_{\rho=0}^{4} = \frac{\pi}{2} \left( \frac{1}{2} \right) 4^3 = 16\pi$$

6. Find the volume of the solid between the cone $z = 2\sqrt{x^2 + y^2}$ and the paraboloid $z = 8 - x^2 - y^2$.
   **HINT:** Find the radius where the cone and paraboloid intersect.

   a. $30\pi$
   b. $\frac{80\pi}{3}$
   c. $\frac{40\pi}{3}$ Correct Choice
   d. $\frac{20\pi}{3}$
   e. $\frac{10\pi}{3}$

   $$z = 2r = 8 - r^2 \quad r^2 + 2r - 8 = 0 \quad (r - 2)(r + 4) = 0 \quad r = 2 \text{ since } r \geq 0.$$

   $$V = \iiint 1 \, dV = \int_0^{2\pi} \int_0^{\sqrt{8 - r^2}} \int_0^r \rho^2 \, dz \, dr \, d\theta = 2\pi \int_0^2 \left[ rz \right]_{z=2r}^{8-r^2} \, dr = 2\pi \int_0^2 r(8 - r^2 - 2r) \, dr$$

   $$= 2\pi \int_0^2 (8r - r^3 - 2r^2) \, dr = 2\pi \left[ 4r^2 - \frac{r^4}{4} - \frac{2r^3}{3} \right]_{r=0}^{r=2} = 2\pi \left( 16 - 4 - \frac{16}{3} \right) = \frac{40\pi}{3}$$

7. Compute $\iiint_S \vec{F} \cdot d\vec{S}$ for the vector field $\vec{F} = (z, z, x+y)$ over the surface $S$ which is parametrized by $\vec{R}(u,v) = (u + v, u - v, uv)$ for $0 \leq u \leq 2$ and $0 \leq v \leq 3$ and oriented along $\vec{N} = \vec{e}_u \times \vec{e}_v$.

   a. $-60$
   b. $-12$
   c. $0$
   d. $12$ Correct Choice
   e. $60$

   $$\vec{R}(u,v) = (u + v, u - v, uv) \quad \vec{e}_u = \left( \begin{array}{c} 1 \\ 1 \\ v \end{array} \right) \quad \vec{e}_v = \left( \begin{array}{c} 1 \\ -1 \\ u \end{array} \right)$$

   $$\vec{N} = \vec{e}_u \times \vec{e}_v = i(u+v) - j(u-v) + k(-1 - 1) = (u + v, v - u, -2)$$

   $$\vec{F} = (z, z, x+y) \quad \vec{F}(\vec{R}(u,v)) = (uv, uv, 2u) \quad \vec{F} \cdot \vec{N} = uv(u+v) + uv(v-u) + 2u(-2) = u(2v^2 - 4)$$

   $$\iiint_S \vec{F} \cdot d\vec{S} = \iint S \vec{F}(\vec{R}(u,v)) \cdot \vec{N} \, du \, dv = \int_0^3 \int_0^2 u(2v^2 - 4) \, du \, dv = \left[ \frac{u^2}{2} \right]_0^2 \left[ \frac{2v^3}{3} - 4v \right]_0^3 = (2)(18 - 12) = 12$$
8. Compute \( \iint_C \vec{F} \cdot d\vec{S} \) for the vector field \( \vec{F} = (xz, yz, z^2) \) over the cylindrical surface \( x^2 + y^2 = 9 \) for \( 0 \leq z \leq 2 \) oriented outward.

   a. \( 18\pi \)
   b. \( \frac{62}{3} \pi \)
   c. \( 21\pi \)
   d. \( \frac{124}{3} \pi \)
   e. \( 36\pi \) Correct Choice

\[
\vec{R}(\theta, z) = (3\cos \theta, 3\sin \theta, z) \\
\vec{e}_\theta = \left( -3\sin \theta, 3\cos \theta, 0 \right) \\
\vec{e}_z = (0, 0, 1) \\
\vec{N} = \vec{e}_\theta \times \vec{e}_z = \hat{i}(3\cos \theta) - \hat{j}(3\sin \theta) + \hat{k}(0) = (3\cos \theta, 3\sin \theta, 0) \\
\vec{F} = (xz, yz, z^2) \\
\vec{F}(\vec{R}(\theta, z)) = (3z\cos \theta, 3z\sin \theta, z^2) \\
\vec{F} \cdot \vec{N} = 9z\cos^2 \theta + 9z\sin^2 \theta = 9z \\
\iint_C \vec{F} \cdot d\vec{S} = \iint \vec{F}(\vec{R}(\theta, z)) \cdot \vec{N} \, d\theta \, dz = \int_0^{2\pi} \int_0^2 9z \, d\theta \, dz = 9(2\pi) \left[ \frac{z^2}{2} \right]_0^2 = 36\pi
\]

Work Out: (Points indicated. Part credit possible. Show all work.)

9. (5 points) At the right is the contour plot of a function \( f(x, y) \). If you start at the dot at \((5, 6)\) and move so that your velocity is always in the direction of \( \nabla f \), the gradient of \( f \), roughly sketch your path on the plot.

NOTE: The numbers on the right are the values of \( f \) on each level curve.

The curve starts at \((5, 6)\) goes down and curves to the right towards higher values of the function \( f \), always perpendicular to each level curve. It should not go up.
10. (15 points) An aquarium in the shape of a rectangular solid has a base made of marble which costs 6 cents per square inch, a back and sides made of mirrored glass which costs 2 cents per square inch and a front made of clear glass which costs 1 cent per square inch. There is no top. If the volume of the aquarium is 9000 cubic inches, what are the dimensions of the cheapest such aquarium?

Let \(x\) be the length side to side, \(y\) be the width front to back, and \(z\) be the height.

The cost is

\[C = 6xy + 2(2xz + 2yz) + 1xz = 6xy + 3xz + 4yz\]

The volume constraint is

\[V = xyz = 9000\]

which we solve for \(z = \frac{9000}{xy}\)

So the cost becomes

\[C = 6xy + \frac{27000}{y} + \frac{36000}{x}\]

We want to minimize \(C\). So we set the partials equal to zero:

\[C_x = 6y - \frac{36000}{x^2} = 0\]
\[C_y = 6x - \frac{27000}{y^2} = 0\]

\[y = \frac{6000}{x^2}\]
\[x = \frac{27000}{6y^2} = \frac{4500}{y^2}\]

\[\frac{4500x^4}{6000 \cdot 6000} = \frac{1}{8000}x^4\]

Cancel an \(x\) and solve for \(x\) then \(y\) and \(z\):

\[1 = \frac{x^3}{8000}, \quad x = 20\]
\[y = \frac{6000}{x^2} = \frac{6000}{400} = 15\]
\[z = \frac{9000}{xy} = \frac{9000}{300} = 30\]

So the dimensions are:

\[x = 20, \quad y = 15, \quad z = 30\]

11. (15 points) Compute \(\iiint xy\,dA\) over the "diamond" shaped region bounded by the curves

\[y^2 - x^2 = 4\] \(y^2 - x^2 = 16\]
\[y^2 - 2x^2 = 1\] \(y^2 - 2x^2 = 4\)

**HINT:** Let \(u = y^2 - x^2\) and \(v = y^2 - 2x^2\).

\[u - v = y^2 - x^2 - y^2 + 2x^2 = x^2\]
\[2u - v = 2y^2 - 2x^2 - y^2 + 2x^2 = y^2\]

\[x = \sqrt{u - v}\]
\[y = \sqrt{2u - v}\]

\[J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{array} \right| = \left| \begin{array}{cc} 1 & 2 \\ -1 & 2 \end{array} \right| = \frac{1}{4u - v \sqrt{2u - v}} \]

\[x = \sqrt{u - v}, \quad y = \sqrt{2u - v}\]

\[4 \leq u \leq 16, \quad 1 \leq v \leq 4\]

\[\iiint xy\,dA = \int_{1}^{4} \int_{16}^{1} \frac{1}{4u - v \sqrt{2u - v}} du dv = \int_{1}^{4} \frac{1}{4}(16 - 4)(4 - 1) = 9\]
12. (15 points) Find the average temperature on the hemisphere surface \( x^2 + y^2 + z^2 = 9 \), \( z \geq 0 \), if the temperature is \( T = z \).

**NOTE**: The average of a function \( f \) is \( f_{\text{ave}} = \frac{\iint f \, dS}{\iint dS} \). **HINT**: Parametrize the hemisphere.

\[
\vec{R}(\phi, \theta) = (3 \sin \phi \cos \theta, 3 \sin \phi \sin \theta, 3 \cos \phi) \quad \vec{e}_\phi = (3 \cos \phi \cos \theta, 3 \cos \phi \sin \theta, -3 \sin \phi) \quad \vec{e}_\theta = (-3 \sin \phi \sin \theta, 3 \sin \phi \cos \theta, 0)
\]

\[
\vec{N} = \vec{e}_\phi \times \vec{e}_\theta = i(9 \sin^2 \phi \cos \theta) - j(-9 \sin^2 \phi \sin \theta) + k(9 \sin \phi \cos \phi \cos^2 \theta + 9 \sin \phi \cos \phi \sin^2 \theta) = (9 \sin^2 \phi \cos \theta, 9 \sin^2 \phi \sin \theta, 9 \sin \phi \cos \phi)
\]

\[
|\vec{N}| = \sqrt{81 \sin^4 \phi \cos^2 \theta + 81 \sin^4 \phi \sin^2 \theta + 81 \sin^2 \phi \cos^2 \phi} = 9 \sin \phi
\]

\[
\iint dS = \int_0^{2\pi} \int_0^{\pi/2} |\vec{N}| \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/2} 9 \sin \phi \, d\phi \, d\theta = 18\pi \quad \text{(Area of hemisphere) = \( \frac{1}{2}(4\pi R^2) \).}
\]

\[
\iint T \, dS = \frac{1}{18\pi} \cdot 27\pi = \frac{27\pi}{18} = \frac{3}{2}
\]

13. (15 points) Sketch the region of integration and then compute the integral \( \int_0^1 \int_{x^2}^1 x^3 \cos(y^3) \, dy \, dx \).

Reverse the order of integration:

\[
\int_0^1 \int_{x^2}^1 x^3 \cos(y^3) \, dy \, dx = \int_0^1 \int_0^{y^3} x^3 \cos(y^3) \, dx \, dy = \int_0^1 \left[ \frac{x^4}{4} \cos(y^3) \right]_{x=0}^{x=y^3} \, dy = \int_0^1 \frac{y^2}{4} \cos(y^3) \, dy
\]

Substitute: \( u = y^3 \quad du = 3y^2 \, dy \quad y^2 \, dy = \frac{1}{3} \, du \)

\[
\int_0^1 x^3 \cos(y^3) \, dx \, dy = \frac{1}{12} \int \cos(u) \, du = \frac{1}{12} \sin(u) \bigg|_0^1 = \frac{1}{12} \sin 1
\]