Multiple Choice: (8 points each) Work Out: (points indicated)

1. Find the volume of the parallelepiped with edges $\vec{u} = (1,0,3)$, $\vec{v} = (0,2,-1)$ and $\vec{w} = (2,0,2)$.
   
   a. $-8$
   b. $-4$
   c. 4
   d. 8 correct choice
   e. 16

   \[ V = |\vec{u} \cdot \vec{v} \times \vec{w}| = \begin{vmatrix} 1 & 0 & 3 \\ 0 & 2 & -1 \\ 2 & 0 & 2 \end{vmatrix} = |-8| = 8 \]

2. Duke Skywater is flying the Millenium Eagle through a polaron field. His galactic coordinates are $\text{ (2300,4200,1600) }$ measured in lightseconds and his velocity is $\vec{v} = (.2,.3,.4)$ measured in lightseconds per second. He measures the strength of the polaron field is $p = 274$ milliwookies and its gradient is $\vec{V}p = (3,2,2)$ milliwookies per lightsecond. Assuming a linear approximation for the polaron field and that his velocity is constant, how many seconds will Duke need to wait until the polaron field has grown to $286$ milliwookies?

   a. 2
   b. 3
   c. 4
   d. 6 correct choice
   e. 12

   The derivative along Duke’s path is

   \[ \frac{dp}{dt} = \vec{v} \cdot \vec{V}p = (.2,.3,.4) \text{ lightseconds/second} \cdot (3,2,2) \text{ milliwookies/lightsecond} = .6 + .6 + .8 = 2 \text{ milliwookies/second} \]

   So the polaron field increases 2 milliwookies each second.
   To increase 12 milliwookies, it will take 6 seconds.
3. Find the plane tangent to the hyperbolic paraboloid \( x = yz \) at the point \( P = (6,3,2) \). Which of the following points does **not** lie on this plane?

a. \((-6,0,0)\)
b. \((0,3,0)\)
c. \((0,0,2)\)
d. \((-1,1,1)\)
e. \((1,-1,-1)\) correct choice

Let \( f(y,z) = yz \). Then \( f_y = z \) and \( f_z = y \).

At the point \((y,z) = (3,2)\), we have \( f(3,2) = 6, f_y(3,2) = 2 \) and \( f_z(3,2) = 3 \).

So the plane tangent to \( x = f(y,z) \) at \((y,z) = (3,2)\) is

\[
x = f_{\text{tan}}(y,z) = f(3,2) + f_y(3,2)(y-3) + f_z(3,2)(z-2) = 6 + 2(y-3) + 3(z-2)
\]

or \( x = 2y + 3z - 6 \)

Plugging in each point, we find \((1,-1,-1)\) is not a solution.

4. A airplane is circling with constant speed above Kyle Field along the curve

\( \vec{r}(t) = (\cos(8\pi t), \sin(8\pi t), 2) \) where distances are in miles and time is in hours.

Find the tangential acceleration \( a_T \), where the acceleration is \( \vec{a} = a_T \hat{T} + a_N \hat{N} \).

a. 0 correct choice
b. \( 8\pi \)
c. \(-8\pi \)
d. \( 64\pi^2 \)
e. \(-64\pi^2 \)

\[
\vec{v} = (-8\pi \sin(8\pi t), 8\pi \cos(8\pi t), 0) \quad \vec{a} = (-64\pi^2 \cos(8\pi t), -64\pi^2 \sin(8\pi t), 0)
\]

\[
|\vec{v}| = \sqrt{64\pi^2} = 8\pi \quad \hat{T} = (-\sin(8\pi t), \cos(8\pi t), 0) \quad a_T = \frac{d|\vec{v}|}{dt} = 0
\]

OR \( a_T = \vec{a} \cdot \hat{T} = 64\pi^2 \cos(8\pi t) \sin(8\pi t) - 64\pi^2 \cos(8\pi t) \sin(8\pi t) = 0 \)

5. Find the volume below the plane \( z = 6 - 2y \) above the triangle with vertices \((0,0,0)\), \((1,0,0)\) and \((0,3,0)\).

a. 3
b. 6 correct choice
c. 9
d. 12
e. 15

\[
V = \int_0^1 \int_0^{3-3x} (6 - 2y) \, dy \, dx = \int_0^1 [6y - y^2]_0^{3-3x} \, dx = \int_0^1 [6(3 - 3x) - (3 - 3x)^2] \, dx = \int_0^1 (9 - 9x^2) \, dx
\]

\[
= [9x - 3x^3]_0^1 = 9 - 3 = 6
\]
6. (10 points) Find the location and value of the minimum of the function 
\( f(x, y, z) = x^2 + 2y^2 + 3z^2 \) on the plane \( x + y + z = 11 \).

METHOD 1: Lagrange Multipliers:
\[
\begin{align*}
\nabla f &= (2x, 4y, 6z) \\
g &= x + y + z \\
\nabla g &= (1, 1, 1) \\
\lambda \nabla f &= \nabla g \\
2x &= \lambda, \quad 4y = \lambda, \quad 6z = \lambda \\
x &= \frac{\lambda}{2}, \quad y = \frac{\lambda}{4}, \quad z = \frac{\lambda}{6} \\
x + y + z &= \frac{\lambda}{2} + \frac{\lambda}{4} + \frac{\lambda}{6} = 11 \\
\lambda &= 12 \\
x &= 6, \quad y = 3, \quad z = 2 \\
f(6, 3, 2) &= 36 + 2 \cdot 9 + 3 \cdot 4 = 66
\end{align*}
\]

METHOD 2: Eliminate a Variable:
\( z = 11 - x - y \) \Rightarrow \( f = x^2 + 2y^2 + 3(11 - x - y)^2 \)
\( f_x = 2x - 6(11 - x - y) = 8x + 6y - 66 = 0 \) \quad \( f_y = 4y - 6(11 - x - y) = 6x + 10y - 66 = 0 \)
Cross multiply: \( 80x + 60y = 660 \) \quad \( 36x + 60y = 396 \) \Rightarrow \( 44x = 264 \) \Rightarrow \( x = 6 \)
Substitute back into \( f_y \): \( 36 + 10y - 66 = 0 \) \Rightarrow \( y = 3 \)
Substitute back: \( z = 11 - 6 - 3 = 2 \) \quad \( f(6, 3, 2) = 36 + 2 \cdot 9 + 3 \cdot 4 = 66 \)

7. (10 points) Consider the region between the curves 
\( y = 2|x| - 2 \) and \( y = |x| \).
If the density is \( \delta = 2 + 2y \) compute the mass and \( y \)-component of the center of mass of this region.
(7 points for setup. 3 points for evaluation.)

Find positive intersection: \( 2x - 2 = x \) \Rightarrow \( x = 2 \)
Use symmetry to double the integral for positive \( x \).
\[
M = 2 \int_0^2 \int_{2x-2}^x (2 + 2y) \, dy \, dx = 2 \int_0^2 [2y + y^2]_{2x-2}^x \, dx = 2 \int_0^2 [2x + x^2] - [2(2x - 2) + (2x - 2)^2] \, dx \\
= 2 \int_0^2 (6x - 3x^2) \, dx = 2[3x^2 - x^3]_0^2 = 2(12 - 8) = 8
\]
\[y\text{-mom} = 2 \int_0^2 \int_{2x-2}^x y(2 + 2y) \, dy \, dx = 2 \int_0^2 \left[y^2 + \frac{2y^3}{3}\right]_0^x \, dx \\
= 2 \int_0^2 \left[x^2 + \frac{2x^3}{3}\right] - \left[(2x - 2)^2 + \frac{2(2x - 2)^3}{3}\right] \, dx \\
= 2 \int_0^2 \left(\frac{4}{3} - 8x + 13x^2 - \frac{14}{3}x^3\right) \, dx = 2 \left[\frac{4}{3}x - 4x^2 + 13x^3 - \frac{14}{3}x^4\right]_0^2 \\
= 2\left(\frac{8}{3} - 16 + \frac{104}{3} - \frac{56}{3}\right) = \frac{16}{3}(1 - 6 + 13 - 7) = \frac{16}{3}
\]
\[\bar{y} = \frac{y\text{-mom}}{M} = \frac{16}{3} \cdot \frac{3}{8} = \frac{2}{3}\]
8. (20 points) **Stokes’ Theorem** states that if \( S \) is a nice surface in \( \mathbb{R}^3 \) and \( \partial S \) is its boundary curve traversed counterclockwise as seen from the tip of the normal to \( S \) and \( \vec{F} \) is a nice vector field on \( S \) then

\[
\oint_S \nabla \times \vec{F} \cdot dS = \int_{\partial S} \vec{F} \cdot d\vec{s}
\]

Verify Stokes’ Theorem if

\[ F = (y, -x, x^2 + y^2) \]

and \( S \) is the paraboloid \( z = x^2 + y^2 \) for \( z \leq 4 \) with **normal pointing up and in**.

Remember to check the orientations.

The paraboloid may be parametrized by:

\[ \vec{R}(r, \theta) = (r \cos \theta, r \sin \theta, r^2) \]

a. (10) Compute \( \oint_S \nabla \times \vec{F} \cdot d\vec{S} \) using the following steps:

\[
\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y & -x & x^2 + y^2 \end{vmatrix} = i(2y - 0) - j(2x - 0) + k(-1 - 1) = (2y, -2x, -2)
\]

\[
\left( \nabla \times \vec{F} \right) \left( \vec{R}(r, \theta) \right) = (2r \sin \theta, -2r \cos \theta, -2)
\]

\[ \vec{R}_r = (\cos \theta, \sin \theta, 2r) \]
\[ \vec{R}_\theta = (-r \sin \theta, r \cos \theta, 0) \]

\[ \vec{N} = i(-2r^2 \cos \theta) - j(2r^2 \sin \theta) + k(r \cos^2 \theta + r \sin^2 \theta) = (-2r^2 \cos \theta, -2r^2 \sin \theta, r) \]

This is oriented correctly as up and in.

\[
\oint_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_C \nabla \times \vec{F} \cdot \vec{N} \, dr \, d\theta = \oint_C (-4r^3 \sin \theta \cos \theta + 4r^3 \sin \theta \cos \theta - 2r) \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^{-2r} (-2r) \, dr \, d\theta = 2\pi \left[ -r^2 \right]_0^{-2} = -8\pi
\]

b. (10) Recall \( F = (y, -x, x^2 + y^2) \) and \( S \) is the paraboloid \( z = x^2 + y^2 \) for \( z \leq 4 \) with **normal pointing up and in**. Compute \( \oint_{\partial S} \vec{F} \cdot d\vec{s} \) using the following steps:

\[ \vec{r}(\theta) = (2 \cos \theta, 2 \sin \theta, 4) \]
\[ \vec{v}(\theta) = (-2 \sin \theta, 2 \cos \theta, 0) \]

which is correctly counterclockwise.

\[ \vec{F}(\vec{r}(\theta)) = (2 \sin \theta, -2 \cos \theta, 4) \]

\[
\oint_{\partial S} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v} \, d\theta = \int_0^{2\pi} (-4 \sin^2 \theta - 4 \cos^2 \theta) \, d\theta = \int_0^{2\pi} (-4) \, d\theta = -8\pi
\]
9. (10 points) The paraboloid at the right is the graph of the equation \( z = x^2 + y^2 \).
It may be parametrized as
\[
\vec{R}(r, \theta) = (r \cos \theta, r \sin \theta, r^2).
\]
Find the area of the paraboloid for \( z \leq 4 \).
HINT: Use results from #8.
\[
\vec{R}_r = (\cos \theta, \sin \theta, 2r) \\
\vec{R}_\theta = (-r \sin \theta, r \cos \theta, 0) \\
\vec{N} = i(-2r^2 \cos \theta) - j(2r^2 \sin \theta) + k(r \cos^2 \theta + r \sin^2 \theta) = (-2r^2 \cos \theta, -2r^2 \sin \theta, r) \\
|\vec{N}| = \sqrt{4r^4 \cos^2 \theta + 4r^4 \sin^2 \theta + r^2} = \sqrt{4r^4 + r^2} = r\sqrt{4r^2 + 1} \\
A = \iint |\vec{N}| \, dr \, d\theta = \int_0^{2\pi} \int_0^2 r\sqrt{4r^2 + 1} \, dr \, d\theta = 2\pi \left[ \frac{(4r^2 + 1)^{3/2}}{3} \right]_0^2 = \frac{\pi}{6}(17^{3/2} - 1)
\]

10. (10 points) A paraboloid in \( \mathbb{R}^4 \) with coordinates \((w, x, y, z)\), may be parametrized by
\( (w, x, y, z) = \vec{R}(r, \theta) = (r \cos \theta, r \sin \theta, r^2, r^2) \) for \( 0 \leq r \leq 3 \) and \( 0 \leq \theta \leq 2\pi \).
Compute \( I = \iint (xz \, dw \, dy - wy \, dx \, dz) \) over the surface.
\[
w = r \cos \theta, \quad x = r \sin \theta, \quad y = r^2, \quad z = r^2 \\
\frac{\partial(w, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ 2r & 0 \end{vmatrix} = 2r^2 \sin \theta \quad \frac{\partial(x, z)}{\partial(r, \theta)} = \begin{vmatrix} \sin \theta & r \cos \theta \\ 2r & 0 \end{vmatrix} = -2r^2 \cos \theta \\
I = \int_0^{2\pi} \int_0^3 (r^3 \sin \theta(2r^2 \sin \theta) - r^3 \cos \theta(-2r^2 \cos \theta)) \, dr \, d\theta = \int_0^{2\pi} \int_0^3 2r^5 \, dr \, d\theta = 2\pi \left[ \frac{r^6}{3} \right]_0^3 = 486\pi
\]