Multiple Choice: (7 points each)

1. Consider the line through the point \( P = (4, 4, 4) \) which is perpendicular to the plane \( x + 2y + 3z = 7 \). Its tangent vector is
   \[ \text{a. } (3, 2, 1) \quad \text{b. } (1, 2, 3) \quad \text{c. } (7, 6, 5) \quad \text{d. } (5, 6, 7) \quad \text{e. } (4, 4, 4) \]

If the equation of a plane is \( Ax + By + Cz = D \) then the normal is \( \vec{N} = (A, B, C) \). In this case, \( \vec{N} = (1, 2, 3) \). Since the line is perpendicular to the plane, then its tangent vector is the normal to the plane. So, \( \vec{v} = (1, 2, 3) \).

2. Find the plane tangent to the hyperbolic paraboloid \( x - yz = 0 \) at the point \( P = (6, 3, 2) \). Which of the following points does not lie on this plane?
   \[ \text{a. } (-6, 0, 0) \quad \text{b. } (0, 3, 0) \quad \text{c. } (0, 0, 2) \quad \text{d. } (1, -1, -1) \quad \text{e. } (-1, 1, 1) \]

The hyperbolic paraboloid is a level surface of the function \( g = x - yz \). Its gradient is \( \nabla g = (1, -z, -y) \). So the normal to the surface at \( P \) is \( \vec{N} = \nabla g \big|_{(6,3,2)} = (1, -2, -3) \). So the tangent plane is \( \vec{N} \cdot X = \vec{N} \cdot P \), or \( x - 2y - 3z = 6 - 2 \cdot 3 - 3 \cdot 2 = -6 \). Plugging in each point, we find \((1, -1, -1)\) is not a solution.

3. Duke Skywater is flying the Millenium Eagle through a polaron field. His galactic coordinates are \( (2300, 4200, 1600) \) measured in lightseconds and his velocity is \( \vec{v} = (.2, .3, .4) \) measured in lightseconds per second. He measures the strength of the polaron field is \( p = 274 \) milliwookies and its gradient is \( \nabla p = (3, 2, 2) \) milliwookies per lightsecond. Assuming a linear approximation for the polaron field and that his velocity is constant, how many seconds will Duke need to wait until the polaron field has grown to \( 286 \) milliwookies?
   \[ \text{a. } 2 \quad \text{b. } 3 \quad \text{c. } 4 \quad \text{d. } 6 \quad \text{e. } 12 \]
The derivative along Duke’s path is

\[
\frac{dp}{dt} = \mathbf{v} \cdot \nabla p = (0.2, 0.3, 0.4, 0.6 \text{ lightseconds}) \cdot (3.2, 2.2, 0.8 \text{ milliwookies per lightsecond})
\]

\[
= 0.6 + 0.6 + 0.8 = 2 \text{ milliwookies per second}
\]

So the polaron field increases 2 milliwookies each second. To increase 12 milliwookies, it will take 6 seconds.

4. Consider the surface \( S \) parametrized by \( \mathbf{r}(u, v) = (u + v, u - v, uv) \) for \( 0 \leq u \leq 2 \) and \( 0 \leq v \leq 4 \). Compute \( \iint_S \mathbf{F} \cdot d\mathbf{S} \) where \( \mathbf{F} = (y, x, y) \).

a. -32
b. -16
c. 16 correct choice
d. 32
e. 64

\[
\mathbf{r}_u = (1, 1, v) \quad \mathbf{r}_v = (1, -1, u) \quad \mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = (u + v, v - u, -2)
\]

\[
\mathbf{F} = (y, x, y) = (u - v, u + v, u - v)
\]

\[
\mathbf{F} \cdot \mathbf{N} = (u - v)(u + v) + (u + v)(v - u) - 2(u - v) = -2u + 2v
\]

\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{N} du dv = \int_0^4 \int_0^4 (-2u + 2v) du dv = \int_0^4 \left[ -u^2 + 2uv \right]_{u=0}^{u=2} dv
\]

\[
= \int_0^4 \left[ -4 + 4v \right] dv = \left[ -4v + 2v^2 \right]_0^4 = -16 + 32 = 16
\]

5. Consider the surface \( S \) parametrized by \( \mathbf{r}(u, v) = (u + v, u - v, uv) \). Find the plane tangent to this surface at the point \( P = \mathbf{r}(1, 2) = (3, -1, 2) \). Which of the following points does not lie on this plane?

a. (3, 0, 0) correct choice
b. (0, 4, 0)
c. (0, 0, -2)
d. (1, 1, 0)
e. (0, 6, 1)

\[
\mathbf{r}_u = (1, 1, v) \quad \mathbf{r}_v = (1, -1, u) \quad \mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = (u + v, v - u, -2)
\]

The normal at \( P \) is \( \mathbf{N}_P = \mathbf{N}_{(1,2)} = (3, 1, -2) \) and the tangent plane is \( \mathbf{N} \cdot \mathbf{X} = \mathbf{N} \cdot \mathbf{P} \), or

\[
3x + y - 2z = 3(3) + (-1) - 2(2) = 4.
\]

Plugging in each point, we find (3, 0, 0) is not a solution.
6. Compute \( \iint (-x^2y^2 \, dx + 2xy^3 \, dy) \) over the complete boundary of the semicircular area \( 0 \leq y \leq \sqrt{4-x^2} \) traversed counterclockwise.

a. 0 
b. 16 
c. \( \frac{4}{5} \) 
d. \( \frac{80}{5} \) 
e. \( \frac{128}{5} \) correct choice

By Green’s Theorem:
\[
\iint (-x^2y^2 \, dx + 2xy^3 \, dy) = \iint \left( \frac{\partial}{\partial x} (2xy^3) - \frac{\partial}{\partial y} (-x^2y^2) \right) \, dx \, dy = \iint (2y^3 + 2x^2y) \, dx \, dy
\]
\[
= \left[ \int_0^\pi \int_0^2 2r^2 \sin \theta \, r \, dr \, d\theta \right] = 2 \left[ -\cos \theta \right]_0^\pi \left[ \frac{r^5}{5} \right]_0^2 = \frac{128}{5}
\]

7. Compute \( \iiint s \frac{x^2z^2}{3} \, dy \, dz + \frac{y^2z^2}{3} \, dz \, dx + \frac{z^5}{5} \, dx \, dy \) over the complete surface of the sphere \( x^2 + y^2 + z^2 = 4 \) with outward normal.

a. \( \frac{512\pi}{21} \) correct choice 
b. \( \frac{32\pi^2}{4} \) 
c. \( \frac{128\pi}{5} \) 
d. \( \frac{16\pi}{3} \) 
e. \( \frac{256\pi}{15} \)

Apply Gauss’ Theorem in spherical coordinates:
\[
\vec{F} = \left( \frac{x^2z^2}{3}, \frac{y^2z^2}{3}, \frac{z^5}{5} \right) \quad \vec{\nabla} \cdot \vec{F} = x^2z^2 + y^2z^2 + z^4 = (x^2 + y^2 + z^2)z^2 = \rho^2 \cdot \rho^2 \cos^2 \theta
\]
\[
I = \iiint \vec{\nabla} \cdot \vec{F} \, dV = \int_0^{2\pi} \int_0^\pi \int_0^2 \rho^4 \cos^2 \theta \cdot \rho^2 \sin \theta \, d\rho \, d\phi \, d\theta = 2\pi \left[ -\frac{\cos^3 \theta}{3} \right]_0^\pi \left[ \frac{\rho^7}{7} \right]_0^2 = \frac{512\pi}{21}
\]

8. (15 points) Find the point in the first octant on the surface \( z = \frac{32}{x^4y^2} \) which is closest to the origin.

Minimize \( f = x^2 + y^2 + z^2 \) on the surface \( g = zx^4y^2 = 32 \).
\[
\vec{\nabla} f = (2x, 2y, 2z) \quad \vec{\nabla} g = (4xz^3y^2, 2zx^4y, x^4y^2) \quad \vec{\nabla} f = \lambda \vec{\nabla} g
\]
\[
2x = \lambda 4xz^3y^2 \quad 2y = \lambda 2zx^4 \quad 2z = \lambda x^4y^2 \quad \lambda = \frac{1}{2zx^2y^2} = \frac{1}{zx^4} = \frac{2z}{x^4y^2}
\]
\[
x^2 = 2y^2 \quad y^2 = 2z^2 \quad x = \sqrt{2}y \quad z = \frac{1}{\sqrt{2}}y
\]
\[
g = zx^4y^2 = \left( \frac{1}{\sqrt{2}}y \right) \left( \sqrt{2}y \right)^4 y^2 = 2^{32}y^7 = 32 = 2^5 \quad y^7 = 2^{7/2}
\]
\[
y = \sqrt{2} \quad x = 2 \quad z = 1 \quad (x, y, z) = (2, \sqrt{2}, 1)
9. (10 points) Compute \( \iint_R x \, dA \) over the region \( R \) in the first quadrant bounded by the curves 
\[ y = x^2, \quad y = x^4 \quad \text{and} \quad y = 16. \]

The left edge is \( y = x^4 \) or \( x = y^{1/4} \). The right edge is \( y = x^2 \) or \( x = y^{1/2} \).

\[
\iint_R x \, dA = \int_1^{16} \int_{x^{1/4}}^{x^{1/2}} x \, dy \, dx = \int_1^{16} \left[ \frac{x^2}{2} \right]_{x^{1/4}}^{x^{1/2}} \, dy = \int_1^{16} \left[ \frac{y}{2} - \frac{y^{1/2}}{2} \right] \, dy \\
= \left[ \frac{x^2}{4} - \frac{y^{3/2}}{3} \right]_1^{16} = \left[ \frac{256}{4} - \frac{64}{3} \right] - \left[ \frac{1}{4} - \frac{1}{3} \right] = \frac{255}{4} - 21 = \frac{171}{4}
\]

10. (15 points) Find the mass and center of mass of the solid below the paraboloid 
\( z = 4 - x^2 - y^2 \) above the \( xy \)-plane, if the density is \( \delta = x^2 + y^2 \). (11 points for setting up the integrals and the final formula.)

In cylindrical coordinates, the paraboloid is \( z = 4 - r^2 \), the density is \( \delta = r^2 \) and the Jacobian is \( r \).

\[
M = \iiint \delta \, dV = \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} r^2 \, dz \, dr \, d\theta \\
= 2\pi \int_0^2 \int_0^{4-r^2} r^2 \, dz \, dr = 2\pi \int_0^2 r^3 (4 - r^2) \, dr \\
= 2\pi \left[ r^4 - \frac{r^6}{6} \right]_0^2 = 2\pi \left( 16 - \frac{32}{3} \right) = \frac{32\pi}{3}
\]

\( z \)-mom = \( \iiint z \delta \, dV = \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} rz^3 \, dz \, dr \, d\theta \\
= 2\pi \int_0^2 r^3 \frac{z^2}{2} \left|_0^{4-r^2} \right. \, dr = \pi \int_0^2 r^3 (4 - r^2)^2 \, dr \\
\text{Let } u = r^2. \text{ Then } du = 2r \, dr \text{ and } r \, dr = \frac{1}{2} \, du. \text{ So} \\
z \text{-mom } = \frac{\pi}{2} \int_0^4 u(4-u)^2 \, du = \frac{\pi}{2} \int_0^4 u(16 - 8u + u^2) \, du = \frac{\pi}{2} \left[ 8u^2 - 8\frac{u^3}{3} + \frac{u^4}{4} \right]_0^4 \\
= \frac{\pi}{2} \left( 128 - \frac{512}{3} + 64 \right) = \frac{32\pi}{3}
\]

\[
\bar{z} = \frac{z \text{-mom}}{M} = \frac{32\pi}{3} \cdot \frac{3}{32\pi} = 1
\]

\[
\bar{x} = \bar{y} = 0 \quad \text{by symmetry.}
\]
11. (15 points) Find the area and centroid of the right leaf of the rose

\[ r = 2 \cos^2 \theta. \]

(12 points for setting up the integrals and the final formula.)

\[
A = \int_{-\pi/2}^{\pi/2} \int_{0}^{2 \cos^2 \theta} r \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \left[ \frac{r^2}{2} \right]_{0}^{2 \cos^2 \theta} d\theta = \int_{-\pi/2}^{\pi/2} 2 \cos^4 \theta d\theta = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \left( 1 + 2 \cos 2\theta + \frac{1}{2} \cos 4\theta \right) d\theta
\]

\[
= \frac{3}{4} \left( \frac{\pi}{2} - \frac{-\pi}{2} \right) = \frac{3\pi}{4}
\]

\[
x\text{-mom} = \int_{-\pi/2}^{\pi/2} \int_{0}^{2 \cos^2 \theta} r^2 \cos \theta \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \left[ \frac{r^3}{3} \right]_{0}^{2 \cos^2 \theta} \cos \theta d\theta = \frac{8}{3} \int_{-\pi/2}^{\pi/2} \cos^6 \theta \cos \theta d\theta
\]

\[
= \frac{8}{3} \int_{-\pi/2}^{\pi/2} (1 - \sin^2 \theta)^3 \cos \theta d\theta
\]

\[
= \frac{8}{3} \int_{-\pi/2}^{\pi/2} (1 - \frac{1}{2} u^2)^3 \, du = \frac{8}{3} \int_{-1}^{1} (1 - \frac{3}{2} u^2 + 3u^4 - u^6) \, du
\]

\[
= \frac{8}{3} \left[ u - \frac{3}{5} u^5 - \frac{u^7}{7} \right]_{-1}^{1} = \frac{16}{3} \left[ 1 - \frac{3}{5} - \frac{1}{7} \right] = \frac{16}{3} \frac{16}{35} = \frac{256}{105}
\]

\[
\bar{x} = \frac{x\text{-mom}}{A} = \frac{256}{105} \cdot \frac{4}{3\pi} = \frac{1024}{315\pi}
\]

\[
\bar{y} = 0 \quad \text{by symmetry.}
\]