1. (10 points) Let $P_1$ be the vector space of polynomials of degree $\leq 1$. Suppose $L : P_1 \to \mathbb{R}$ is a linear map which satisfies

$$L(2 + 3t) = 1, \quad L(1 + 4t) = -2.$$ 

Compute $L(5 - 2t)$.

2. (10 points) Which of the following is not a subspace of $C^1[-1, 1]$? Why?

$$P = \{ f \in C^1[-1, 1] \mid f(-1) = f(1) \} \quad Q = \left\{ f \in C^1[-1, 1] \mid \frac{f(-1) + f(1)}{2} = f(0) \right\}$$

$$R = \left\{ f \in C^1[-1, 1] \mid \int_0^1 f(t) \, dt = 1 \right\} \quad S = \left\{ f \in C^1[-1, 1] \mid f'(0) = f(0) \right\}$$
3. (10 points) Duke Skywater is flying the Millennium Eagle through the Asteroid Belt. At the current time, his position is \( \overrightarrow{r} = (4, -1, 2) \) and his velocity is \( \overrightarrow{v} = (3, 2, -1) \). He measures that the electric field and its Jacobian are currently

\[
\vec{E} = \begin{pmatrix} 12 \\ 2 \\ 9 \end{pmatrix} \quad \text{and} \quad J\!E = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 3 \\ 1 & 2 & 9 \end{pmatrix}.
\]

Use a linear (affine) approximation to estimate what the electric field will be 2 sec from now.

4. (10 points) Let \( L : \mathbb{R}^5 \to \mathbb{R}^4 \) be a linear map whose matrix is \( A \). If \( A \) is row reduced, one obtains the matrix

\[
\begin{pmatrix} 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

What is the dimension of the kernel of \( L \)? What is the dimension of the image of \( L \)? Be sure to explain why.
5. (60 points) Let $M_{2,2}$ be the vector space of $2 \times 2$ matrices. Let $P_2$ be the vector space of polynomials of degree $\leq 2$. Consider the linear map $L : M_{2,2} \rightarrow P_2$ given by

$$L(M) = \left( \begin{array}{c} 1 \\ x \end{array} \right) M \left( \begin{array}{c} 1 \\ x \end{array} \right)$$

Hint: For some parts it may be useful to write $M = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ and/or $p(x) = a + \beta x + \gamma x^2$.

a. (3) Identify the domain of $L$, a basis for the domain, and the dimension of the domain.

b. (3) Identify the codomain of $L$, a basis for the codomain, and the dimension of the codomain.

c. (6) Identify the kernel of $L$, a basis for the kernel, and the dimension of the kernel.

d. (6) Identify the image of $L$, a basis for the image, and the dimension of the image.
e. (2) Is the function $L$ one-to-one? Why?

f. (2) Is the function $L$ onto? Why?

g. (2) Verify the dimensions in a, b, c and d agree with the Nullity-Rank Theorem.

h. (6) Find the matrix of $L$ relative to the standard bases: (Call it $A$.)

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for } M(2,2)$$

and $E_1 = 1, \quad E_2 = x, \quad E_3 = x^2$ for $P_2$
i. (6) Another basis for $P_2$ is $F_1 = 1 + x$, $F_2 = 1 + x^2$, $F_3 = x + x^2$. Find the change of basis matrices between the $E$ and $F$ bases. (Call them $C_{F\rightarrow E}$ and $C_{E\rightarrow F}$.) Be sure to identify which is which!

j. (6) Consider the polynomial $q = 2 + 4x$. Find $[q]_E$ and $[q]_F$, the components of $q$ relative to the $E$ and $F$ bases, respectively. Check $[q]_F$. 
k. (5) Find the matrix of $L$ relative to the $e$ basis for $M(2,2)$ and the $F$ basis for $P_2$. (Call it $B_{F \to e}$.)

l. (5) Find $B_{F \to e}$ by a second method.
m. (6) Consider the matrix \[ N = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \]. Find \([N]_E\), the components of \(N\) relative to the \(E\) basis and \([L(N)]_F\), the components of \(L(N)\) relative to the \(F\) basis. Use \([L(N)]_F\) to find \(L(N)\)?

n. (2) Recompute \(L(N)\) using the definition of \(L\).