Multiple Choice: (10 points each)  Work Out: (15 points each)  Extra Credit: (10 points)

1. If \( \vec{V} \times \vec{F} = (2x^2y - x^2, y^2 - 2xy^2, 2xz - 2yz) \), then \( \vec{V} \cdot (\vec{V} \times \vec{F}) = \)
   
   a. \( (4xy - 2x, 4xy - 2y, 2x - 2y) \)  
   b. \( (4xy - 2x, 2y - 4xy, 2x - 2y) \)  
   c. \( 8xy - 4y \)  
   d. 0  
   e. \( 2x^2 - 2y^2 \)

For any vector field \( \vec{V} \cdot \vec{V} \times \vec{F} = 0 \)

2. If \( \vec{V} \times \vec{F} = (2x^2y - x^2, y^2 - 2xy^2, 2xz - 2yz) \), then \( \vec{V} \times (\vec{V} \times \vec{F}) = \)
   
   a. \( (4xy - 2x, 4xy - 2y, 2x - 2y) \)  
   b. \( (4xy - 2x, 2y - 4xy, 2x - 2y) \)  
   c. \( (-2z, -2z, -2x^2 - 2y^2) \)  
   d. \( (-2z, 2z, -2x^2 - 2y^2) \)  
   e. \( -2x^2 - 2y^2 \)

\[
\vec{V} \times \vec{V} \times \vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
2x^2y - x^2 & y^2 - 2xy^2 & 2xz - 2yz
\end{vmatrix} = i(-2z) - j(2z) + k(-2y^2 - 2x^2)
\]

3. If \( \vec{G} = (2x^2y - x^2, y^2 - 2xy^2, 2xz - 2yz) \), then \( \vec{G} = \vec{V}g \) where \( g(0, 1, 1) - g(0, 1, 0) = \)
   
   a. -2  
   b. -1  
   c. 0  
   d. 1  
   e. The scalar potential \( g \) does not exist.  

Since \( \vec{V} \times \vec{G} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
2x^2y - x^2 & y^2 - 2xy^2 & 2xz - 2yz
\end{vmatrix} = i(-2z) - j(2z) + k(-2y^2 - 2x^2) \neq (0, 0, 0), \)

the scalar potential \( g \) does not exist.
4. Compute \( \int (y + z) \, dx + (x + z) \, dy + (x + y) \, dz \) clockwise around the circle \( x^2 + y^2 = 9 \) with \( z = 5 \).
   
   HINT: Use a theorem.
   
   a. \(-18\pi\)
   b. \(-9\pi\)
   c. 0 correct choice
   d. \(9\pi\)
   e. \(18\pi\)

   Let \( \vec{F} = (y + z, x + z, x + y) \). Then \( \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + z & x + z & x + y \end{vmatrix} = (0, 0, 0) \).

   By Stokes' Theorem, \( \oint \vec{F} \cdot d\vec{s} = \iint \nabla \times \vec{F} \cdot d\vec{s} = 0 \)

   where the surface integral is over the disk bounded by the circle.

5. Compute \( \oint (xy) \, dx + (xy) \, dy \) counterclockwise around the boundary of the triangle with vertices \((0, 0), (1, 0)\) and \((0, 2)\).

   By Green's theorem,

   \[
   \oint (xy) \, dx + (xy) \, dy = \iint_T \left( \frac{\partial (xy)}{\partial x} - \frac{\partial (xy)}{\partial y} \right) \, dx \, dy
   \]

   \[
   = \int_0^1 \int_{2-2x}^{2-x} (y - x) \, dy \, dx = \int_0^1 \left[ \frac{y^2}{2} - xy \right]_{2-2x}^{2-x} \, dx
   \]

   \[
   = \int_0^1 \left[ \left( \frac{2 - 2x}{2} \right)^2 - x(2 - 2x) \right] \, dx = \int_0^1 \left( 2 - 4x + 2x^2 - 2x + 2x^2 \right) \, dx
   \]

   \[
   = \int_0^1 \left( 2 - 6x + 4x^2 \right) \, dx = \left[ 2x - 3x^2 + 4\frac{x^3}{3} \right]_0^1 = \left[ 2 - 3 + \frac{4}{3} \right] = \frac{1}{3}
   \]
6. Compute \( \iint_{\partial P} \vec{E} \cdot d\vec{S} \) for \( \vec{E} = (xz, yz, z^2) \) over the complete surface of the solid paraboloid \( P \) given by \( x^2 + y^2 \leq z \leq 4 \) with outward normal.

By Gauss’ Theorem, \( \iint_{\partial P} \vec{E} \cdot d\vec{S} = \iiint_P \nabla \cdot \vec{E} \, dV = \iiint_P (z + z + 2z) \, dV = \iiint_P (4z) \, dV \)

In cylindrical coordinates, the paraboloid is \( r^2 \leq z \leq 4 \). So

\[
\iint_{\partial P} \vec{E} \cdot d\vec{S} = \int_0^{2\pi} \int_0^4 (4z) r \, dz \, d\theta \, dr = 2\pi \int_0^2 [2z^2]_r^4 \, r \, dr = 2\pi \int_0^2 (32 - 2r^4) \, r \, dr \\
= 2\pi \left[ 32 \frac{r^2}{2} - 2 \frac{r^6}{6} \right]_0^2 = 2\pi \left( 64 - \frac{64}{3} \right) = 128\pi \left( 1 - \frac{1}{3} \right) = \frac{256\pi}{3}
\]

7. The cone \( z = \sqrt{x^2 + y^2} \) for \( z \leq 2 \)
is shown at the right. Find the mass and center of mass of the cone if its surface density is given by \( \delta = x^2 + y^2 \).

\( \vec{R}(r, \theta) = (r \cos \theta, r \sin \theta, r) \quad \vec{e}_r = (\cos \theta, \sin \theta, 1) \quad \vec{e}_\theta = (-r \sin \theta, r \cos \theta, 0) \)

\( \vec{N} = \vec{e}_r \times \vec{e}_\theta = (-r \cos \theta, -r \sin \theta, r) \quad |\vec{N}| = \sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta + r^2} = r \sqrt{2} \)

\( \delta = x^2 + y^2 = r^2 \)

\( M = \iint_C \delta \, dS = \iint_C \delta(\vec{R}(r, \theta)) \, |\vec{N}| \, dr \, d\theta = \int_0^{2\pi} \int_0^2 r^2 \sqrt{2} \, r \, dr \, d\theta = 2\pi \sqrt{2} \frac{r^4}{4} \bigg|_0^2 = 8\pi \sqrt{2} \)

By symmetry, \( \bar{x} = \bar{y} = 0 \).

\( z\text{-mom} = \iint_C z \delta \, dS = \iint_C z \delta(\vec{R}(r, \theta)) \, |\vec{N}| \, dr \, d\theta = \int_0^{2\pi} \int_0^2 r^3 \sqrt{2} \, r \, dr \, d\theta = 2\pi \sqrt{2} \frac{r^5}{5} \bigg|_0^2 = \frac{64\pi \sqrt{2}}{5} \)

\( \bar{z} = \frac{z\text{-mom}}{M} = \frac{64\pi \sqrt{2}}{5 \cdot 8\pi \sqrt{2}} = \frac{8}{5} \)
8. Compute \( \int \int_S \nabla \times \vec{F} \cdot d\vec{S} \) for \( \vec{F} = (x^2y, -x^3, z^2) \) over the piece of the sphere \( x^2 + y^2 + z^2 = 25 \) for \( 0 \leq z \leq 4 \) with normal pointing away from the z-axis.

Hint: Parametrize the upper and lower edges.

By Stokes’ Theorem \( \int \int_S \nabla \times \vec{F} \cdot d\vec{S} = \oint \vec{F} \cdot d\vec{S} = \oint_{\text{upper}} \vec{F} \cdot d\vec{S} + \oint_{\text{lower}} \vec{F} \cdot d\vec{S} \)

By the right hand rule, since the normal points outward, the upper circle must be traversed clockwise while the lower circle must be traversed counterclockwise as seen from the positive z-axis. We compute each line integral:

**Upper Circle:** \( z = 4 \) \( x^2 + y^2 = 25 - z^2 = 25 - 16 = 9 \)

\[ \vec{r}(t) = (3 \cos t, 3 \sin t, 4) \quad \vec{v} = (-3 \sin t, 3 \cos t, 0) \]

This is clockwise, so we reverse the velocity: \( \vec{v} = (3 \sin t, -3 \cos t, 0) \)

\[ \vec{F} = (x^2y, -x^3, z^2) = (27 \cos^2 t \sin t, -27 \cos^3 t, 16) \]

\[ \oint_{\text{upper}} \vec{F} \cdot d\vec{S} = \oint_{\text{upper}} \vec{F} \cdot \vec{v} \, dt = \int_0^{2\pi} (81 \cos^2 t \sin^2 t + 81 \cos^4 t) \, dt = 81 \int_0^{2\pi} \cos^2 t \sin^2 t + \cos^2 t \, dt \]

\[ = 81 \int_0^{2\pi} \frac{1 + \cos 2t}{2} \, dt = \frac{81}{2} \left[ t + \frac{\sin 2t}{2} \right]_0^{2\pi} = 81\pi \]

**Lower Circle:** \( z = 0 \) \( x^2 + y^2 = 25 - z^2 = 25 \)

\[ \vec{r}(t) = (5 \cos t, 5 \sin t, 0) \quad \vec{v} = (-5 \sin t, 5 \cos t, 0) \]

This is counterclockwise, so we do not need to reverse the velocity.

\[ \vec{F} = (x^2y, -x^3, z^2) = (125 \cos^2 t \sin t, -125 \cos^3 t, 0) \]

\[ \oint_{\text{lower}} \vec{F} \cdot d\vec{S} = \oint_{\text{lower}} \vec{F} \cdot \vec{v} \, dt = \int_0^{2\pi} (-625 \cos^2 t \sin^2 t - 625 \cos^4 t) \, dt = -625 \int_0^{2\pi} \cos^2 t \sin^2 t + \cos^2 t \, dt \]

\[ = -625 \int_0^{2\pi} \frac{1 + \cos 2t}{2} \, dt = -625 \left[ t + \frac{\sin 2t}{2} \right]_0^{2\pi} = -625\pi \]

**Total Boundary:**

\[ \oint \vec{v} \times \vec{F} \cdot d\vec{S} = 81\pi - 625\pi = -544\pi \]
Extra Credit  Redo #8 but compute \( \iiint_S \nabla \times \vec{F} \cdot d\vec{S} \) directly as a surface integral.

Use spherical coordinates:

\[ \vec{R}(\varphi, \theta) = (5 \sin \varphi \cos \theta, 5 \sin \varphi \sin \theta, 5 \cos \varphi) \quad \arccos \frac{4}{5} \leq \varphi \leq \frac{\pi}{2} \quad 0 \leq \theta \leq 2\pi \]

\[ \vec{e}_\varphi = (5 \cos \varphi \cos \theta, 5 \cos \varphi \sin \theta, -5 \sin \varphi) \quad \vec{e}_\theta = (-5 \sin \varphi \sin \theta, 5 \sin \varphi \cos \theta, 0) \]

\[ \vec{N} = \vec{e}_\varphi \times \vec{e}_\theta = (25 \sin^2 \varphi \cos \theta, 25 \sin^2 \varphi \sin \theta, 25 \sin \varphi \cos \varphi) \]

\[ \vec{N} \cdot \vec{F} = -2500 \sin^3 \varphi \cos \varphi \cos^2 \theta \]

\[ \iiint_S \vec{N} \cdot \vec{F} \, d\vec{S} = \iiint_S \nabla \times \vec{F} \cdot \vec{N} \, d\varphi \, d\theta = \int_0^{2\pi} \int_{\arccos 4/5}^{\pi/2} -2500 \sin^3 \varphi \cos \varphi \cos^2 \theta \, d\varphi \, d\theta \]

\[ = -2500 \int_0^{2\pi} \cos^2 \theta \, d\theta \int_{\arccos 4/5}^{\pi/2} \sin^3 \varphi \cos \varphi \, d\varphi = -2500 \left[ \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{2\pi} \left[ \frac{\sin^4 \varphi}{4} \right]_{\arccos 4/5}^{\pi/2} \]

\[ = -\frac{2500}{4} [\pi] [1 - \sin^4 \arccos 4/5] = -\frac{2500\pi}{4} \left[ 1 - \left( \frac{3}{5} \right)^4 \right] = -\frac{2500\pi}{4} \left( \frac{625 - 81}{625} \right) = -544\pi \]