1. Consider an ideal gas whose density, $\rho$, temperature, $T$, and pressure, $P$, are functions of position. Thus if we consider a two dimensional space $\mathbb{R}^2$ whose coordinates are $(\rho, T)$ then the ideal gas law, $P = k\rho T$, defines a function $P : \mathbb{R}^2 \rightarrow \mathbb{R}$. (Here $k$ is a constant which may appear in your answers.) Further, the formulas which give $(\rho, T)$ as functions of position $(x, y, z)$ define a function $F : \mathbb{R}^3 \rightarrow \mathbb{R}^2$. The composition $P \circ F : \mathbb{R}^3 \rightarrow \mathbb{R}$ then gives $P$ as a function of position. At the point $X = (2, 3, 4)$, $\rho$, $T$ and their gradients are

$$\rho(X) = 2 \quad T(X) = 78 \quad \vec{\nabla} \rho(X) = (0.1, 0.2, -0.1) \quad \vec{\nabla} T(X) = (0.2, -0.3, 0.4)$$

a. What is $JF(X) = \frac{d(P, T)}{d(x, y, z)}(X)$, the Jacobian matrix of $F$ at $X$?

b. What are $JP$ and $JP(\rho(X), T(X))$, the Jacobian matrix of $P$ and the Jacobian matrix of $P$ at $X$?

c. What is $J(P \circ F)(X)$, the Jacobian matrix of $P \circ F$ at $X$?
d. Use the linear approximation to estimate \( P(Y) \), the pressure at the point \( Y = (2.2, 2.9, 4.1) \).

e. At the time \( t = 0 \), you are at \( X = (2, 3, 4) \) and moving with velocity \( \vec{v} = (-1, 1, 2) \).
Use the linear approximation to estimate the temperature \( T \) at time \( t = 2 \).

The remainder of the exam is customized for each student.
Exam 1 #2: Find the non-parametric equation for the plane tangent to the surface 
\[ x^3y^2 + xz^3 = 31 \] at the point \((x,y,z) = (1,2,3)\).
Exam 1 #4: Let \( M = \begin{pmatrix} 2 & 5 & 4 & -1 \\ 0 & 1 & -2 & 1 \\ 1 & 3 & 0 & -2 \\ 2 & 6 & 3 & x \end{pmatrix} \)

a. Compute \( \det M \) (as a function of \( x \)).

b. For what value(s) of \( x \) does \( M^{-1} \) exist? Why?
Exam 1 #5: Let $A = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$.

a. Compute $A^{-1}$. Check it.

b. Solve the equations

$$3x + 2y = 2$$
$$x - z = 1$$
$$y + 2z = 3$$
Exam 1 #6: (Multiple Choice: Circle one) If $C = AB$, then $(C^T)^{-1} =$

a. $A^TB^{-1} + A^{-1}B^T$

b. $B^{-1}A^T + B^TA^{-1}$

c. $(A^{-1})^T(B^{-1})^T$

d. $(B^T)^{-1}(A^T)^{-1}$

Now prove it. You may use any result proved in class or in the book or on homework.
Exam 2 #1: Let \((P_2)^2\) be the vector space of ordered pairs of polynomials of degree less than 2. For example,
\[
\vec{q} = \left(\frac{2x - 3}{3x + 1}\right) \in (P_2)^2 \quad \text{and} \quad \vec{q}(2) = \left(\frac{1}{7}\right)
\]
The standard basis of \((P_2)^2\) is
\[
e_1 = \left(\begin{array}{c} 1 \\ 0 \end{array}\right) \quad e_2 = \left(\begin{array}{c} x \\ 0 \end{array}\right) \quad e_3 = \left(\begin{array}{c} 0 \\ 1 \end{array}\right) \quad e_4 = \left(\begin{array}{c} 0 \\ x \end{array}\right)
\]
Another basis for \((P_2)^2\) is
\[
E_1 = \left(\begin{array}{c} 1 \\ 0 \end{array}\right) \quad E_2 = \left(\begin{array}{c} 1 + x \\ 0 \end{array}\right) \quad E_3 = \left(\begin{array}{c} 0 \\ 1 \end{array}\right) \quad E_4 = \left(\begin{array}{c} 0 \\ 1 + x \end{array}\right)
\]

a. Find the change of basis matrices \(C\) and \(C\).

b. Find \((\vec{q})_e\) the components of \(\vec{q} = \left(\frac{2x - 3}{3x + 1}\right)\) relative to the \(e\)-basis.

c. Find \((\vec{q})_E\) the components of \(\vec{q}\) relative to the \(E\)-basis by using the change of basis matrix.

d. If \((\vec{r})_E = \left(\begin{array}{c} \frac{1}{2} \\ \frac{2}{3} \\ \frac{3}{4} \end{array}\right)\), what is \(\vec{r}\)?
 Exam 2 #2: Let $(P_2)^2$ be the vector space of ordered pairs of polynomials of degree less than 2. For example,

$$\vec{q} = \left( \frac{2x - 3}{3x + 1} \right) \in (P_2)^2 \quad \text{and} \quad \vec{q}(2) = \left( \frac{1}{7} \right)$$

Consider the subspace $S$ of $(P_3)^2$ spanned by $\left( \frac{1 + x}{1 - x} \right)$, $\left( \frac{2 + x}{2 - x} \right)$, $\left( \frac{3 + x}{3 - x} \right)$, $\left( \frac{1 - x}{1 + x} \right)$. Pare the spanning set down to a basis for $S$ and find the dimension of $S$. 
Exam 2 #3: Let \((P_2)^2\) be the vector space of ordered pairs of polynomials of degree less than 2. For example,

\[ \vec{q} = \begin{pmatrix} 2x - 3 \\ 3x + 1 \end{pmatrix} \in (P_2)^2 \quad \text{and} \quad \vec{q}(2) = \begin{pmatrix} 1 \\ 7 \end{pmatrix} \]

The standard basis of \((P_2)^2\) is

\[ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} x \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad e_4 = \begin{pmatrix} 0 \\ x \end{pmatrix} \]

Another basis for \((P_2)^2\) is

\[ E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 + x \\ 0 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad E_4 = \begin{pmatrix} 0 \\ 1 + x \end{pmatrix} \]

The change of basis matrices are

\[ C_{e \to E} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad C_{E \to e} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

Now consider the linear map \(L: (P_2)^2 \to P_2\) given by \(L(\vec{p}) = p_1 + p_2\). (Just add the two component polynomials.) For example, if \( \vec{q} = \begin{pmatrix} -3 + 2x \\ 1 + 3x \end{pmatrix} \), then

\[ L(\vec{q}) = L\left( \begin{pmatrix} -3 + 2x \\ 1 + 3x \end{pmatrix} \right) = (-3 + 2x) + (1 + 3x) = -2 + 5x \]

a. Find the matrix of \(L\) relative to the \(e\)-basis on \((P_2)^2\) and the \(f\)-basis on \(P_2\) where \(f_1 = 1\) and \(f_2 = x\). Call it \(A\).

b. Find the matrix of \(L\) relative to the \(E\)-basis on \((P_2)^2\) and the \(f\)-basis on \(P_2\) by using the change of basis matrix. Call it \(B\).
c. Find the matrix of $L$ relative to the $E$-basis on $(P_2)^2$ and the $f$-basis on $P_2$ from the definition.

d. If $(\vec{r}_E) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, what are $[L(\vec{r})]_f$ and $L(\vec{r})$?
Exam 2 #4: Let \((P_2)^2\) be the vector space of ordered pairs of polynomials of degree less than 2. For example,

\[
\vec{q} = \left( \frac{2x - 3}{3x + 1} \right) \in (P_2)^2 \quad \text{and} \quad \vec{q}(2) = \left( \frac{1}{7} \right)
\]

Consider the linear map \( L : (P_2)^2 \to P_2 \) given by \( L(\vec{p}) = p_1 + p_2 \). When necessary, let

\[
\vec{p} = \left( \begin{array}{c} p_1 \\ p_2 \end{array} \right) = \left( \begin{array}{c} a + bx \\ c + dx \end{array} \right).
\]

a. Find the kernel of \( L \). Give a basis and the dimension.

b. Find the image of \( L \). Give a basis and the dimension.

c. Is \( L \) one-to-one? Why?

d. Is \( L \) onto? Why?

e. Check that the Nullity-Rank Theorem is satisfied.
Exam 2 #5: Let \((P_2)^2\) be the vector space of ordered pairs of polynomials of degree less than 2. For example,
\[
\vec{q} = \left( \frac{2x - 3}{3x + 1} \right) \in (P_2)^2 \quad \text{and} \quad \vec{q}(2) = \left( \frac{1}{7} \right)
\]
Verify that the following function is an inner product on \((P_2)^2\):
\[
\langle \ , \ \rangle : (P_2)^2 \times (P_2)^2 \to \mathbb{R} \quad \text{given by} \quad \langle \vec{p}, \vec{q} \rangle = \int_{-1}^{1} p_1(x)q_1(x) + p_2(x)q_2(x) \, dx
\]
For example,
\[
\left\langle \left( \frac{1 + x}{2x} \right), \left( \frac{-x}{2 - x} \right) \right\rangle = \int_{-1}^{1} (1 + x)(-x) + (2x)(2 - x) \, dx = \int_{-1}^{1} (3x - 3x^2) \, dx = -2
\]

a. Symmetric:

b. Bilinear:

c. Positive Definite:
**Exam 2 #6:** Let \((P_2)^2\) be the vector space of ordered pairs of polynomials of degree less than 2. For example,

\[\vec{q} = \left(\frac{2x - 3}{3x + 1}\right) \in (P_2)^2 \quad \text{and} \quad \vec{q}(2) = \left(\frac{1}{7}\right)\]

Using the following inner product on \((P_2)^2\):

\[
\langle \ , \ \rangle : (P_2)^2 \times (P_2)^2 \to \mathbb{R} \quad \text{given by} \quad \langle \vec{p}, \vec{q} \rangle = \int_{-1}^{1} p_1(x)q_1(x) + p_2(x)q_2(x) \, dx
\]

find the angle between the vectors \(\left(\frac{1}{x}\right)\) and \(\left(\frac{1}{-x}\right)\).
Exam 3 #2: Consider the parametric curve \( r(t) = \left(2t, t^2, \frac{1}{3}t^3\right) \) for \( 0 \leq t \leq 2 \).

a. Compute \( \int_{(0,0,0)}^{(4,4,8/3)} (xy + 3z) \, ds \) along this curve.

b. Compute \( \int_{(0,0,0)}^{(4,4,8/3)} \mathbf{F} \cdot \, ds \) along this curve where \( \mathbf{F} = (3z, 2y, x) \).
Exam 3 #3: Consider the parametric surface

\[ \vec{R}(p, q) = (p, q, p^2 + q^2) \]

for \(-1 \leq p \leq 1\) and \(-1 \leq q \leq 1\).

a. Find the total mass \( M = \iint \delta \, dS \) on this surface if the surface density is \( \delta = \sqrt{4z + 1} \).

b. Find the flux \( \iint \vec{F} \cdot d\vec{S} \) of the vector field \( \vec{F} = (3x, 3y, 3z) \) through this surface with normal pointing down.
Exam 3 #4: Use 2 methods to compute
\[ \iint_C \vec{F} \cdot d\vec{S} \quad \text{for} \quad \vec{F} = (5xz, 5yz, z^2) \]
over the conical surface \( C \) given by
\[ z = \sqrt{x^2 + y^2} \leq 3 \]
with normal pointing down and out.

a. METHOD 1: Compute \( \iint_C \vec{F} \cdot d\vec{S} \) directly as a surface integral using the parametrization \( \vec{R}(r, \theta) = (r \cos \theta, r \sin \theta, r) \).

HINT: Find \( \vec{e}_r, \vec{e}_\theta, \vec{N} \) and \( \vec{F} \) on the cone.
Recall: \( \vec{F} = (5xz, 5yz, z^2) \) and \( C \) is the conical surface \( z = \sqrt{x^2 + y^2} \leq 3 \) with normal pointing down and out.

b. METHOD 2: Compute \( \iint_C \vec{F} \cdot d\vec{S} \) by applying Gauss' Theorem

\[
\iiint_V \nabla \cdot \vec{F} \, dV = \iint_{\partial V} \vec{F} \cdot d\vec{S}
\]

to the solid cone \( V \) whose boundary is \( \partial V = C + D \)

where \( C \) is the conical surface and \( D \) is the disk at the top of the cone.