Determinant Theorems

Definitions:
If \( A \) is an \( n \times n \) matrix, then the determinant of \( A \), denoted by either \( \det A \) or \( |A| \), is defined by

\[
\det A = |A| = \sum_{\text{perm } p} \varepsilon_p A_{1p_1} A_{2p_2} \cdots A_{np_n}
\]

where \( \varepsilon_p \) is the sign of the permutation \( p \) given by

\[
\varepsilon_p = \begin{cases} 
1 & \text{if } p \text{ is even} \\
-1 & \text{if } p \text{ is odd}
\end{cases}
\]

Row Notation: If \( \vec{v}_1, \cdots, \vec{v}_n \) are \( n \) vectors in \( \mathbb{R}^n \) then \( \det(\vec{v}_1, \cdots, \vec{v}_n) \) is the determinant of the matrix whose rows are \( \vec{v}_1, \cdots, \vec{v}_n \); i.e.

\[
\det(\vec{v}_1, \cdots, \vec{v}_n) = \det\begin{pmatrix}
\vec{v}_1 & \rightarrow \\
\vdots & \vdots \\
\vec{v}_n & \rightarrow
\end{pmatrix}
\]

Theorems:
1. Transpose:
   \( \det A^T = \det A \)
   - Every theorem below involving rows can be restated in terms of columns.
2. Triangular:
   If \( A \) is triangular (or diagonal), then \( \det A \) is the product of the diagonal entries.
3. Row of zeros:
   \( \det\begin{pmatrix} \vec{v}_1, \cdots, \vec{0}, \cdots, \vec{v}_n \end{pmatrix} = 0 \)
4. Interchange rows:
   \( \det(\vec{v}_1, \cdots, \vec{u}, \cdots, \vec{w}, \cdots, \vec{v}_n) = -\det(\vec{v}_1, \cdots, \vec{w}, \cdots, \vec{u}, \cdots, \vec{v}_n) \)…………………………(Row Operation I)
5. Two equal rows:
   \( \det(\vec{v}_1, \cdots, \vec{u}, \cdots, \vec{u}, \cdots, \vec{v}_n) = 0 \)
6. Multiple of row:
   \( \det(\vec{v}_1, \cdots, c\vec{u}, \cdots, \vec{v}_n) = c \det(\vec{v}_1, \cdots, \vec{u}, \cdots, \vec{v}_n) \)…………………………(Row Operation II)
7. Addition in row:
   \( \det(\vec{v}_1, \cdots, \vec{u} + \vec{w}, \cdots, \vec{v}_n) = \det(\vec{v}_1, \cdots, \vec{u}, \cdots, \vec{v}_n) + \det(\vec{v}_1, \cdots, \vec{w}, \cdots, \vec{v}_n) \)
8. Add multiple of one row to another row:
   \( \det(\vec{v}_1, \cdots, \vec{u} + c\vec{w}, \cdots, \vec{w}, \cdots, \vec{v}_n) = \det(\vec{v}_1, \cdots, \vec{u}, \cdots, \vec{w}, \cdots, \vec{v}_n) \)…………………………(Row Operation III)
9. Multiple of matrix:
   \( \det(cA) = c^n \det A \)
10. Product of Matrices:
    \( \det(AB) = \det A \det B \)
11. Invertibility:
    \( \det A \neq 0 \iff A \) is invertible (non-singular) \iff \( AX = B \) has a unique solution
    \( \det A = 0 \iff A \) is non-invertible (singular) \iff \( AX = B \) has no solution or \( \infty \)-many solutions
12. Determinant of inverse:
    If \( A \) is invertible, then \( \det A^{-1} = \frac{1}{\det A} \)