M412 Practice Problems for Final Exam

1. Solve the PDE

\[ u_t + t^3 u_x = u \]
\[ u(t, 0) = t, \quad t > 0 \]
\[ u(0, x) = 1 - e^{-x}, \quad x > 0. \]

2. Solve the PDE

\[ u_{tt} = e^2 u_{xx}; \quad x > 0, t > 0 \]
\[ u(0, x) = f(x); \quad x > 0 \]
\[ u_t(0, x) = g(x); \quad x > 0 \]
\[ u_x(t, 0) = t; \quad t > 0. \]

3. Solve the PDE

\[ u_{xx} + u_{yy} = 0 \]
\[ u(x, 0) = 0, \quad u(x, 2) = 0 \]
\[ u(0, y) = 0, \quad u(1, y) = 2. \]

4. Solve the PDE

\[ u_t = u_{xx} + e^{-t} \sin 3\pi x \]
\[ u(t, 0) = 0, \quad u(t, 1) = 0 \]
\[ u(0, x) = \sin \pi x. \]

5. For the PDE in Problem 4, find an equilibrium solution and show that it matches the limit as \( t \to \infty \) of your solution to Problem 4.

6. For the PDE

\[ u_t = u_{xx} + t \sin x \]
\[ u_x(t, 0) = -1 \]
\[ u_x(t, \pi) = 0 \]
\[ u(0, x) = \cos x, \]

find the total energy

\[ \int_0^\pi u(t, x) dx. \]

7. Use separation of variables to show that solutions to the quarter-plane problem

\[ u_t = u_{xx}; \quad t > 0, x > 0 \]
\[ u_x(t, 0) = 0 \]
\[ |u(t, +\infty)| \text{ bounded} \]
\[ u(0, x) = f(x) \]
can be written in the form
\[ u(t, x) = \int_0^\infty C(\omega) e^{-\omega^2 t} \cos \omega x d\omega, \]
for some appropriate constant \( C(\omega) \).

8. Use the method of Fourier transforms to solve the first order equation
\[ u_t = u_x \]
\[ u(0, x) = f(x). \]

9. [This question appeared on Exam 3.] Use Fourier’s Theorem to prove that if a function \( f(x) \) is piecewise smooth on an interval \([0, L]\), then the Fourier cosine series for \( f(x) \) converges for all \( x \in (0, L) \) to
(i) \( f(x) \) if \( f \) is continuous at the point \( x \)
(ii) \( \frac{1}{2} (f(x^-) + f(x^+)) \) if \( f \) has a jump discontinuity at the point \( x \)

What does the Fourier cosine series converge to at the endpoints \( x = 0 \) and \( x = L \)?

10. We have seen in the homework that if a function \( f(x) \) is piecewise smooth on an interval \([0, L]\), then the Fourier sine series for \( f(x) \) converges for all \( x \in (0, L) \) to
(i) \( f(x) \) if \( f \) is continuous at the point \( x \)
(ii) \( \frac{1}{2} (f(x^-) + f(x^+)) \) if \( f \) has a jump discontinuity at the point \( x \).

Use this and Problem 9 to prove that if \( f(x) \) is continuous on \([0, L]\) and \( f'(x) \) is piecewise smooth on the same interval, then the Fourier cosine series for \( f(x) \) can be differentiated term by term.

**Solutions**

1. For \( x \geq \frac{t^4}{4} \), we have
\[
\frac{dx}{dt} = t^3; \quad x(0) = x_0 \Rightarrow x(t) = \frac{t^4}{4} + x_0
\]
\[
\frac{du}{dt} = u; \quad u(0) = 1 - e^{-x_0} \Rightarrow u(t) = (1 - e^{-x_0})e^t,
\]
from which we conclude
\[ u(t, x) = (1 - e^{-(x - \frac{t^4}{4})})e^t. \]

For \( x \leq \frac{t^4}{4} \), we have
\[
\frac{dx}{dt} = t^3; \quad x(t_0) = 0 \Rightarrow x(t) = \frac{t^4}{4} - \frac{t^4}{4} t_0
\]
\[
\frac{du}{dt} = u; \quad u(t_0) = t_0 \Rightarrow u(t) = t_0 e^{t-t_0},
\]
from which we conclude
\[ u(t, x) = (t^4 - 4x)^{1/4} e^{-(t^4 - 4x)^{1/4}}. \]
Combining these, 
\[
    u(t, x) = \begin{cases} 
        (t^4 - 4x)^{1/4}e^{-(t^4 - 4x)^{1/4}}, & x \leq \frac{t^4}{4} \\
        (1 - e^{-(x - \frac{t^4}{4})})e^{t}, & x \geq \frac{t^4}{4}.
    \end{cases}
\]

2. We write solutions in the form
\[
    u(t, x) = F(x - ct) + G(x + ct),
\]
where for \( x > 0 \), we have
\[
    F(x) = \frac{1}{2}f(x) - \frac{1}{2c} \int_0^x g(y)dy \\
    G(x) = \frac{1}{2}f(x) + \frac{1}{2c} \int_0^x g(y)dy.
\]
This entirely determines the solution for \( x - ct > 0 \). For \( x - ct < 0 \), we need to evaluate \( F \) at negative numbers. In order to do this, we notice that our final condition gives
\[
    t = F'(-ct) + G'(ct).
\]
Setting \( x = -ct \), we find
\[
    F'(x) = -\frac{x}{c} - G'(-x).
\]
We compute, now,
\[
    \int_0^x F'(y)dy = \int_0^x -\frac{y}{c} - G'(-y)dy \Rightarrow F(x) - F(0) = -\frac{x^2}{2c} + G(-x) - G(0).
\]
It’s clear from our expressions for \( F \) and \( G \) that (assuming our solution is continuous) \( F(0) = G(0) \), from which we conclude
\[
    F(x) = -\frac{x^2}{2c} + G(-x).
\]
In this way, for \( x - ct < 0 \),
\[
    F(x - ct) = -\frac{(x - ct)^2}{2c} + G(ct - x).
\]
We have, then
\[
    u(t, x) = \begin{cases} 
        \frac{1}{2}[f(x - ct) + f(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)dy, & x - ct > 0 \\
        -\frac{(x - ct)^2}{2c} + \frac{1}{2}[f(ct - x) + f(x + ct)] + \frac{1}{2c} \int_0^{x+ct} g(y)dy + \frac{1}{2c} \int_0^{ct-x} g(y)dy, & x - ct < 0.
    \end{cases}
\]

3. Since we have a bounded domain, we proceed by separation of variables, letting \( u(x, y) = X(x)Y(y) \), for which we find
\[
    u_{xx} + u_{yy} = 0 \Rightarrow X''(x)Y(y) + X(x)Y''(y) = 0 \Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda.
\]
Observe here in particular that we have chosen the sign in front of \( \lambda \) so that the variable with both boundary conditions 0 (\( Y \) in this case) will have the standard eigenvalue equation, \( Y'' + \lambda Y = 0 \). We have, \( u(x, 0) = 0 \Rightarrow Y(0) = 0, u(x, 2) = 0 \Rightarrow Y(2) = 0, \) and \( u(0, y) = 0 \Rightarrow X(0) = 0 \). We have, then, the two ODE
\[
    \begin{align*}
    Y'' + \lambda Y &= 0; & Y(0) = 0, Y(2) = 0 \\
    X'' - \lambda X &= 0; & X(0) = 0.
    \end{align*}
\]
For the $Y(y)$ equation, we take $Y(y) = C_1 \cos \sqrt{\lambda} y + C_2 \sin \sqrt{\lambda} y$, and use the boundary conditions to conclude

$$Y_n(y) = \sin \frac{n\pi}{2} y, \quad n = 1, 2, 3, \ldots$$

For $X(x)$, we have

$$X(x) = C_3 \cosh \frac{n\pi}{2} x + C_4 \sinh \frac{n\pi}{2} x,$$

for which our boundary condition $X(0) = 0$ determines $C_3 = 0$, eliminating one constant of integration. We finally have our general expansion for $u(x, y)$,

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{2} x \sin \frac{n\pi}{2} y.$$

Finally, we employ our last boundary condition, $u(1, y) = 2$ to obtain the Fourier sine series

$$2 = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{2} \sin \frac{n\pi}{2} y.$$

We have, then

$$A_n \sinh \frac{n\pi}{2} = \frac{2}{2} \int_0^{2} 2 \sin \frac{n\pi}{2} y \, dy = -4 \left( \frac{n\pi}{2} \right)^2 y \bigg|_0^2 = -4 \frac{n\pi}{2} \left[ (-1)^n - 1 \right],$$

where I have explicitly written the fraction $\frac{1}{2}$ as a reminder that it comes from $\frac{1}{2}$. Our solution is

$$u(x, y) = \sum_{n=1}^{\infty} \frac{1}{n\pi} \left[ (-1)^n - 1 \right] \sinh \frac{n\pi}{2} x \sin \frac{n\pi}{2} y.$$

4. Due to the non-homogeneous term, we must proceed here by eigenfunction expansion. First, we construct eigenfunctions, $X_n(x)$, for the homogeneous problem. Substituting $u(t, x) = T(t)X(x)$ into $u_t = u_{xx}$, and considering our boundary conditions, we determine

$$X'' + \lambda X = 0; \quad X(0) = 0, X(1) = 0,$$

for which we have $X_n(x) = \sin n\pi x$. We now look for a solution as an expansion of these eigenfunctions

$$u(t, x) = \sum_{n=1}^{\infty} c_n(t) \sin n\pi x.$$

Substituting this expansion back into the full non-homogeneous equation, we find

$$\sum_{n=1}^{\infty} \left( c_n'(t) + n^2 \pi^2 c_n(t) \right) \sin n\pi x = e^{-t} \sin 3\pi x.$$

The key observation we make here is that this is simply a Fourier sine series with fancy constants, $B_n = c_n'(t) - n^2 \pi^2 c_n(t)$. Consequently, we have

$$c_n'(t) + n^2 \pi^2 c_n(t) = 2 \int_0^1 e^{-t} \sin(3\pi x) \sin(n\pi x) \, dx = \begin{cases} e^{-t}, & n = 3 \\ 0, & n \neq 3. \end{cases}$$
For initial conditions, we take our initial data

\[ u(0, x) = \sin \pi x \Rightarrow \sin \pi x = \sum_{n=1}^{\infty} c_n(0) \sin n\pi x, \]

for which

\[ c_n(0) = 2 \int_{0}^{1} \sin(\pi x) \sin(n\pi x) dx = \begin{cases} 1, & n = 1 \\ 0, & n \neq 1. \end{cases} \]

We have now an ODE to solve for each \( n = 1, 2, 3, \ldots \), but we observe that if both \( c_0' + n^2c_0(t) \) and \( c_n(0) \) are \( 0 \), then \( c_n(t) \equiv 0 \). In this case, the only two expansion coefficients that are not identically \( 0 \) are \( c_1(t) \) and \( c_3(t) \). For \( c_1(t) \), we have

\[ c_1' + \pi^2 c_1 = 0; \quad c_1(0) = 1 \Rightarrow c_1(t) = e^{-\pi^2 t}. \]

For \( c_3(t) \), we have

\[ c_3' + 9\pi^2 c_3 = e^{-t}; \quad c_3(0) = 0, \]

which we solve by the integrating factor method. (Recall that for a general linear first order equation \( y'(t) + p(t)y(t) = g(t) \), the integrating factor is \( e^{\int p(t)dt} \), where the constant of integration can be dropped.) In this case, the integrating factor is simply \( e^{9\pi^2 t} \), and we have

\[
(e^{9\pi^2 t} c_3)' = e^{9\pi^2 t} e^{-t} \Rightarrow e^{9\pi^2 t} c_3(t) = \frac{1}{9\pi^2} (e^{-t} - e^{-9\pi^2 t}) + C.
\]

According to our initial condition \( c_3(0) = 0 \), we have

\[ C = \frac{1}{1 - 9\pi^2}. \]

We conclude that

\[ c_3(t) = \frac{1}{1 - 9\pi^2} (e^{-9\pi^2 t} - e^{-t}), \]

with then

\[ u(t, x) = e^{-\pi^2 t} \sin(\pi x) + \frac{1}{1 - 9\pi^2} (e^{-9\pi^2 t} - e^{-t}) \sin(3\pi x). \]

5. Our equilibrium equation for \( \tilde{u}(x) \) is

\[ \tilde{u}_{xx} = 0 \]

\[ \tilde{u}(0) = 0 \]

\[ \tilde{u}(1) = 0, \]

which is solved by

\[ \tilde{u}(x) \equiv 0. \]

Taking a limit as \( t \to \infty \) of our solution to Problem 4, we see that they agree.

6. Integrating the full equation, we have

\[ \int_{0}^{\pi} u_t dx = \int_{0}^{\pi} u_{xx} dx + \int_{0}^{\pi} t \sin x dx \Rightarrow \frac{d}{dt} \int_{0}^{\pi} u(t, x) dx = u_x(t, \pi) - u_x(t, 0) - t \cos x \bigg|_{0}. \]

It follows that

\[ \frac{d}{dt} \int_{0}^{\pi} u(t, x) dx = 1 + 2t. \]

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Integrating,
\[ \int_0^\pi u(t, x)dx = t + t^2 + C. \]
In order to find \( C \), we use \( u(0, x) = \cos x \) to compute
\[ \int_0^\pi \cos x dx = C \Rightarrow C = 0. \]
We conclude
\[ \int_0^\pi u(t, x)dx = t + t^2. \]

7. Separate variables with \( u(t, x) = T(t)X(x) \), and set
\[ \frac{T'}{T} = \frac{X''}{X} = -\lambda, \]
from which we have the eigenvalue problem
\[ \begin{align*}
X'' + \lambda X &= 0 \\
X'(0) &= 0 \\
X(+\infty) &\text{ bounded}.
\end{align*} \]
In this case, all \( \lambda \geq 0 \) are eigenvalues, with associated eigenfunctions
\[ X_\lambda(x) = \cos \sqrt{\lambda}x. \]
Since the eigenvalues are continuous, we integrate rather than summing, obtaining a general solution of the form
\[ u(t, x) = \int_0^\infty A(\lambda)e^{-\lambda t} \cos \sqrt{\lambda}x d\lambda. \]
Finally, set \( \omega = \sqrt{\lambda} \) to get
\[ u(t, x) = \int_0^\infty A(\omega^2)e^{-\omega^2 t} \cos \omega x 2\omega d\omega. \]
The stated result follows from the choice
\[ C(\omega) = 2\omega A(\omega^2). \]

8. Taking the Fourier transform of this equation, we have
\[ \begin{align*}
\hat{u}_t &= -i\omega \hat{u} \\
\hat{u}(t, \omega) &= \hat{f}(\omega) e^{-i\omega t}.
\end{align*} \]
Inverting, we compute
\[ u(t, x) = \int_{-\infty}^{+\infty} e^{-i\omega x} \hat{f}(\omega) e^{-i\omega t} d\omega = \int_{-\infty}^{+\infty} e^{-i\omega(x+t)} \hat{f}(\omega) d\omega, \]
where this last expression is the inverse transform of \( \hat{f} \), evaluated at \( x + t \). That is,
\[ u(t, x) = f(x + t). \]
9. Since \( f(x) \) is only defined on the interval \([0, L]\), we are free to extend it in any way we like to the full interval \([-L, L]\), where Fourier’s theorem is valid. We extend it as an even function, so that the extension \( f_E(x) \) is defined by

\[
f_E(x) = \begin{cases} 
  f(x), & 0 \leq x \leq L \\
  f(-x), & -L \leq x \leq 0.
\end{cases}
\]

If \( f(x) \) is piecewise smooth on \([0, L]\), then \( f_E(x) \) is piecewise smooth on \([-L, L]\), and Fourier’s Theorem states that \( f_E \) definitely has a convergent Fourier series,

\[
f_E(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L}.
\]

We now compute \( A_0, A_n, \) and \( B_n \), keeping in mind that \( f_E(x) \) is an even function. We have

\[
A_0 = \frac{1}{2L} \int_{-L}^{L} f_E(x) \, dx = \frac{1}{L} \int_{0}^{L} f_E(x) \, dx
\]

\[
A_n = \frac{1}{L} \int_{-L}^{L} f_E(x) \cos \frac{n\pi x}{L} \, dx = \frac{2}{L} \int_{0}^{L} f_E(x) \cos \frac{n\pi x}{L} \, dx
\]

\[
B_n = \frac{1}{L} \int_{-L}^{L} f_E(x) \sin \frac{n\pi x}{L} \, dx = 0.
\]

In this way, we see that the series for \( f_E(x) \) is a Fourier cosine series that converges on \([-L, L]\). If it converges on \([-L, L]\), it must converge on \([0, L]\), and since \( f(x) \) and \( f_E(x) \) agree there, it converges to \( f(x) \).

Last, since \( f_E(x) \) is an even extension, we have

\[
\lim_{x \to 0^-} f_E(x) = \lim_{x \to 0^+} f_E(x)
\]

\[
\lim_{x \to L^-} f_E(x) = \lim_{x \to L^+} f_E(x),
\]

so that the Fourier cosine series of \( f(x) \) converges at \( x = 0 \) to

\[
\lim_{x \to 0^+} f(x),
\]

and at \( x = L \) to

\[
\lim_{x \to L^-} f(x).
\]

10. First, under these assumptions, \( f(x) \) has a convergent Fourier cosine series (by Problem 9),

\[
f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}.
\]

Moreover, \( f'(x) \) has a convergent sine series

\[
f'(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}.
\]

with

\[
B_n = \frac{2}{L} \int_{0}^{L} f'(x) \sin \frac{n\pi x}{L} \, dx = \frac{2}{L} \left[ f(x) \sin \frac{n\pi x}{L} \right]_0^L - \frac{n\pi}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L} \, dx
\]

\[
= -\frac{n\pi}{L} A_n,
\]

which gives precisely the series that arises by differentiating the Fourier cosine series of \( f(x) \) term by term.