8 Application of integration (cont.)

8.4 Work

- Newton’s Law of Motion:
  \[ F = m \frac{d^2s}{dt^2} \]

- Work (the objects are moved along the x-axis):
  
  \( W := Fd \)  work=force \times distance

  \( W := \int_a^b f(x) \, dx \)

**Note 1** The work for a variable continuous force \( f \) has been introduced as the limit of sum as follows: let \( P \) be a partition of \([a, b]\) by points \( x_i, (i = 1..n) \), and let \( \Delta x_i = x_i - x_{i-1} \); \(|P|\) is defined as the “maximum” (supremum) of the \( \Delta x_i = x_i - x_{i-1} \). Choose any point \( x_i^* \in [x_{i-1}, x_i] \); then the force at the point \( x_i^* \) is \( f(x_i^*) \). Hence we get:

  \[ W \approx \lim_{|P| \to 0} \sum_{i=1}^{\infty} f(x_i^*) \Delta x_i \]

- spring:
  
  \( k \): spring constant

  \( x \): stretching distance beyond its natural length

  \( W = k \int_{a-l}^{b-l} x \, dx \)

  \[ = \frac{k}{2} (b-a)(b+a-2l) \]

  where \( l \) is its natural length, and \( a, b \) are the two stretching lengths.

8.5 Average Value of a function

- Theorem 1 (Mean Value Theorem) If \( f \) is continuous on \([a, b]\), then there exists a number \( c \) in \([a, b]\) such that

  \[ \int_a^b f(x) \, dx = f(c)(b-a) \]

- Definition 3 (Average value of a function \( f \)) The average of a function \( f \) on the interval \([a, b]\) is defined as

  \[ f_{\text{ave}} := \frac{1}{b-a} \int_a^b f(x) \, dx \]
8.6 Differential Equations

- **Definition 4 (separable equation)** A separable equation is a first order differential equation that can be written in the form:

  \[
  \frac{dy}{dx} = g(x) \cdot f(y)
  \]

  Solution: divide by \( f(y) \), multiply by \( dx \) and integrate on both sides; hence we get:

  \[
  \int \frac{dy}{f(y)} = \int g(x) \, dx
  \]

- **Definition 5 (logistic growth)** The logistic differential equation is a first order differential equation of the form:

  \[
  \frac{dy}{dx} = ky(M - y).
  \]

  The solution of this differential equation with the initial value \( y(0) = y_0 \) is of the form

  \[
  y(t) = \frac{y_0M}{y_0 + (M - y_0)e^{kt}}.
  \]

**Note 2** For this particular model of population growth, we assumed that the population cannot exceed the maximal size \( M \) at which it consumes its entire food supply.

The solution of this differential equation satisfies

\[
\lim_{t \to \infty} y(t) = M,
\]

which satisfies our assumption.

8.7 First Order Linear Equations

- **Definition 6 (first order linear)** A first order linear differential equation is a differential equation that can be put into the form

  \[
  \frac{dy}{dx} + R(x)y = Q(x)
  \]

  where \( P \) and \( Q \) are continuous functions on a given interval.

The solution of a first order linear differential equation is based on the idea of the integrating factor.

- **Definition 7 (integrating factor)** The integrating factor of a first order linear differential equation of the form (1) is defined as

  \[
  I(x) = e^{\int P(x) \, dx}
  \]

  The solution of (1) is obtained by multiplication on both sides of the equation by the integrating factor and integration on both sides:

  \[
  y(x) = \frac{\int Q(x)I(x) \, dx + C}{I(x)}
  \]
8.8 Arc Length

- the collection of the arc length formula can be found in the following theorem:

**Theorem 2 (the arc length formulas)**  
1. If \( f' \) is continuous on \([a, b]\), then the length of the curve \( y = f(x) \), \( a \leq x \leq b \), is

\[
L = \int_a^b \sqrt{1 + (f'(x))^2} \, dx = \int_a^b \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx. \tag{2}
\]

2. If \( g' \) is continuous on \([c, d]\), then the length of the curve \( g(y) = x \), \( c \leq y \leq d \), is

\[
L = \int_c^d \sqrt{1 + (g'(y))^2} \, dy = \int_c^d \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy. \tag{3}
\]

- **Definition 8 (arc length function)** If a smooth curve \( C \) has the equation \( y = f(x) \), \( a \leq x \leq b \), let \( s(x) \) be the distance along \( C \) from the initial point \( P_0(a, f(a)) \) to the point \( Q(x, f(x)) \). Then \( s \) is a function, called the arc length function, and, by formula (2)

\[
s(x) = \int_a^x \sqrt{1 + (f'(t))^2} \, dt. \tag{4}
\]

A similar formula for the arc length can be derived using formula (3).

8.9 Area of a Surface of Revolution

- **Definition 9 (rotation about the x-axis)** Let \( f \) be a positive function with continuous derivative, we define the surface area of the surface obtained by rotating the curve \( y = f(x) \), \( a \leq x \leq b \), about the \( x \)-axis:

\[
S := 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} \, dx = 2\pi \int_a^b f(x) \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx.
\]

If the curve is described as \( g(y) = x \), \( c \leq y \leq d \), then the formula for the surface becomes

\[
S := 2\pi \int_c^d y \sqrt{1 + (g'(y))^2} \, dy = 2\pi \int_c^d y \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy.
\]

- **Definition 10 (rotation about the y-axis)** Let \( f \) be a positive function with continuous derivative, we define the surface area of the surface obtained by rotating the curve \( y = f(x) \), \( c \leq x \leq d \), about the \( x \)-axis:

\[
S := 2\pi \int_c^d x \sqrt{1 + (f'(x))^2} \, dx = 2\pi \int_c^d x \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx.
\]

If the curve is described as \( g(y) = x \), \( c \leq y \leq d \), then the formula for the surface becomes

\[
S := 2\pi \int_c^d g(y) \sqrt{1 + (g'(y))^2} \, dy = 2\pi \int_c^d g(y) \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy.
\]

Using the notation

\[
ds = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \quad \text{or} \quad ds = \sqrt{1 + \left( \frac{dx}{dy} \right)^2},
\]

where \( y = f(x) \) or \( g(y) = x \) respectively, the formulas for the surface area for surface of revolution are reduced to:
• rotation about the \(x\)-axis:
\[ S := 2\pi \int y \, ds \]

• rotation about the \(y\)-axis:
\[ S := 2\pi \int x \, ds. \]

8.10 Moments and Centers of Mass

• The case: finitely many point masses \((m_i, x_i, y_i)\) in the plane where \(m_i\) is the mass located at the point \((x_i, y_i)\) \((1 \leq i \leq n)\).

  - **Definition 11 (moment of the system about the \(x\)-axis)** The moment of the system about the \(x\) axis is defined as:
    \[ M_x = \sum_{i=1}^{n} m_i y_i \]

  - **Definition 12 (moment of the system about the \(y\)-axis)** The moment of the system about the \(y\) axis is defined as:
    \[ M_y = \sum_{i=1}^{n} m_i x_i \]

  - **Definition 13 (center of mass)** The center of mass \((\bar{x}, \bar{y})\) is given by the formulas:
    \[ \bar{x} = \frac{M_y}{m} \]
    \[ \bar{y} = \frac{M_x}{m} \]
    where \(m = \sum_{i=1}^{n} m_i\) is the total mass.

  **Note 3** The center of mass \((\bar{x}, \bar{y})\) is the point where a single particle of mass \(m\) would have the same moments as the system.

• The case: bounded region \(R\) bounded by the curves \(y = f(x), \ x = a\) and \(x = b\) where \(a < b\).

  - **Definition 14 (moment of the system about the \(x\)-axis)** The moment of the system about the \(x\) axis is defined as:
    \[ M_x = \frac{1}{2} \rho \int_{a}^{b} [f(x)]^2 \, dx \]

  - **Definition 15 (moment of the system about the \(y\)-axis)** The moment of the system about the \(y\) axis is defined as:
    \[ M_y = \rho \int_{a}^{b} x f(x) \, dx \]

  - **Definition 16 (center of mass)** The center of mass \((\bar{x}, \bar{y})\) is given by the formulas:
    \[ \bar{x} = \frac{1}{A} \int_{a}^{b} x f(x) \, dx \]
    \[ \bar{y} = \frac{1}{2A} \int_{a}^{b} [f(x)]^2 \, dx \]
    where \(A\) is the area of the bounded region.

• The case: bounded region \(R\) bounded by the curves \(y = f(x), y = g(x), x = a\) and \(x = b\) where \(f(x) \geq g(x), a < b\).
Definition 17 (center of mass) The center of mass \((\bar{x}, \bar{y})\) is given by the formulas:

\[
\bar{x} = \frac{1}{A} \int_a^b x[f(x) - g(x)] \, dx \\
\bar{y} = \frac{1}{2A} \int_a^b \{[f(x)]^2 - [g(x)]^2\} \, dx
\]

where \(A\) is the area of the bounded region.

- **Theorem 3** Let \(R\) be a plane area that lies entirely on one side of a line \(l\). If \(R\) is rotated about \(l\), then the volume of the resulting solid is the product of the area \(A\) of \(R\) and the distance \(d\) traveled by the centroid of \(R\).

8.11 Hydrostatic Pressure and Force

- **Definition 18 (hydrostatic force)** The force \(F\) exerted by the fluid of density \(\rho [kg/m^3]\) on the horizontal plate with area \(A [m^2]\) at a depth \(d [m]\) below the surface, calculated by the formula

\[
F = mg = \rho g Ad,
\]

where \(g\) is the acceleration due to gravity, is called the hydrostatic force.

- **Definition 19 (hydrostatic pressure)** In general, the pressure \(P\) on a plate is defined to be the force per unit area

\[
P = \frac{F}{A}.
\]

Hence we obtain the hydrostatic pressure \(P\) on the plate:

\[
P = \rho gd.
\]

- **Note 4** — The SI unit measuring system is newton per square meter, which is called a pascal (abbreviation: 1N/m² = 1Pa, 1kPa = 1 kilopascal).

  - Using British units, we write \(P = \rho gd = \delta d\), where \(\delta = \rho g\) is the weight density (as opposed to \(\rho\), which is the mass density).

  example:

  - the mass density of water is \(\rho = 1000 kg/m^3\)
  - the weight density of water is \(\delta = 62.5 lb/ft^3\)

**Note 5** An important principle of fluid pressure is the experimentally verified fact that at any point in a liquid the pressure is the same in all directions. Thus the pressure in any direction at a depth \(d\) in a fluid with mass density \(\rho\) is given by

\[
P = \rho gd = \delta d.
\]

This helps us determine the hydrostatic force against a vertical plate or wall or dam in a fluid (see Appendix).
9 Three-Dimensional Analytic Geometry and Vectors

9.1 Three-Dimensional Coordinate Systems

In order to represent points in space, we first choose a fixed point $O$ (the origin) and three direct lines through $O$ that are perpendicular to each other, called the coordinate axes and labeled the $x$-axis, $y$-axis and $z$-axis. Hence a point in this coordinate system can be represented as a triple $(a,b,c)$.

- **Note 6 (the distance formula in three dimensions)** The distance $|P_1P_2|$ between the points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ is

  $$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

- **Note 7 (Equation of a sphere)** An equation of a sphere with center $C(h,k,l)$ and radius $r$ is

  $$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2.$$

In particular, if the center is the origin, then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2.$$

9.2 Vectors

- **Definition 20 (vector)** A two-dimensional vector is an ordered pair $\mathbf{a} = \langle a_1, a_2 \rangle$ of real numbers. A three-dimensional vector is an ordered triple $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ of real numbers. The numbers $a_1$, $a_2$, and $a_3$ are called the components of $\mathbf{a}$.

  - A representation of the vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is a directed line segment $\overline{AB}$ from any point $A(x,y)$ to any point $B(x + a_1, y + a_2)$. A particular representation of $\mathbf{a}$ is the directed line segment $\overline{OP}$ from the origin to the point $P(a_1, a_2)$, and $\langle a_1, a_2 \rangle$ is called the position vector. Likewise, in three dimensions, the vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is the position vector of the point $P(a_1, a_2, a_3)$.

  - The 0-vector is defined as $\mathbf{0} = \langle 0,0 >$ or $\mathbf{0} = \langle 0,0,0 >$ respectively.

- **Note 8** Given the points $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$, the vector $\mathbf{a}$ with representation $\overline{AB}$ is

  $$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

  - If $A(x_1,y_1,z_1) \neq B(x_2,y_2,z_2)$ then $\overline{AB} \neq \overline{BA}$ (orientation!).

- **Definition 21 (magnitude or length)** The magnitude or length of the vector $\mathbf{v}$ is the length of its representation and is denoted by $|\mathbf{v}|$ or $\|\mathbf{v}\|$.

By using the distance formula we get:

**Theorem 4** The length of the two dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}.$$

The length of the three dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$
• **Definition 22 (vector addition)** If \( \mathbf{a} = <a_1, a_2> \) and \( \mathbf{b} = <b_1, b_2> \), then the vector \( \mathbf{a} + \mathbf{b} \) is defined by

\[
\mathbf{a} + \mathbf{b} = <a_1 + b_1, a_2 + b_2>.
\]

Similarly, for three dimensional vectors:

\[
<a_1, a_2, a_3> + <b_1, b_2, b_3> = <a_1 + b_1, a_2 + b_2, a_3 + b_3>.
\]

• **Definition 23 (scalar multiplication)** If \( c \) is a scalar and \( \mathbf{a} = <a_1, a_2> \), then the vector \( c\mathbf{a} \) is defined by

\[
c\mathbf{a} = <ca_1, ca_2>.
\]

Similarly, for three dimensional vectors:

\[
c <a_1, a_2, a_3> = <ca_1, ca_2, ca_3>.
\]

• **Note 9** — Two vectors are called parallel, if \( \mathbf{b} = c\mathbf{a} \) for some scalar \( c \).

— By the difference \( \mathbf{a} - \mathbf{b} \) of two vectors we mean

\[
\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).
\]

• **Definition 24 (n-dimensional vectors)** An \( n \)-dimensional vector is an ordered \( n \)-tuple:

\[
\mathbf{a} = <a_1, a_2, \ldots, a_n>
\]

where \( a_1, a_2, \ldots, a_n \) are real numbers that are called the components of \( \mathbf{a} \). Addition and scalar multiplication are defined in terms of components just as for the case \( n = 2 \) and \( n = 3 \). We denote by \( V_n \) the set of all \( n \)-dimensional vectors.

• **Theorem 5 (properties of vector)** If \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) are vectors in \( V_n \), then we have the following properties:

1. \( \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \)
2. \( \mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c} \)
3. \( \mathbf{a} + \mathbf{0} = \mathbf{a} \)
4. \( \mathbf{a} + (-\mathbf{a}) = \mathbf{0} \)
5. \( c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b} \)
6. \( (c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a} \)
7. \( (cd)\mathbf{a} = c(d\mathbf{a}) \)
8. \( 1\mathbf{a} = \mathbf{a} \)

### Appendix

#### 9.2 Water-Tank-Problems

**9.2.1 Pumping water out of a tank**

The idea of partition (see section 8.4, Note 1) can be used for the tank problem.

**Problem 1** A tank is filled with water (or other liquids). Find the work required to pump the water out of the outlet.
Solution: Assume that the density $\rho$ is constant. The force required to raise a water particle must overcome the force of the gravity $F = mg$.

- choose the coordinate system (zero at the outlet)
- partition $P$ of $[a,b]$ by points $x_i$: $x_i^* \in [x_{i-1},x_i]$

\[ W_i = g\rho x_i^* V_i = g\rho x_i^* Q_i \Delta x_i \]

- find the work for each volume-element $V_i = \Delta x_i Q_i$ ($Q_i$ is the area of the $i$-th slice):

\[ W = \sum_{i=1}^{n} W_i = g\rho \sum_{i=1}^{n} x_i^* Q_i \Delta x_i \]

- add them together:

\[ \bar{W} = \sum_{i=1}^{n} W_i = g\rho \int_{a}^{b} x Q(x) \, dx \]

- limit process:

\[ W := \lim_{||P|| \to 0} \bar{W} = g\rho \int_{a}^{b} x Q(x) \, dx \]

where $Q(x)$ is the area function of the slices, perpendicular to the $x$-axis.

The problem is now reduced to the area function $Q(x)$ of the slices which are obtained by slicing the tank with planes perpendicular to the $x$-axis.

### 9.2.2 Hydrostatic pressure (force) in a tank

We use the same idea as before for the following problem:

**Problem 2** A tank is given, filled with water (or other liquids); a body is at the bottom of the tank. Find the hydrostatic pressure and force against a vertical face of the body.

**Solution:** Assume that the mass density $\rho$ (or equivalent: the weight density $\delta$) is constant. The force against a small particle in the water $F = \rho g A d$ (see section 8.11 for details).

- choose the coordinate system (zero at water surface level)
- partition $P$ of $[a,b]$ by points $x_i$: $x_i^* \in [x_{i-1},x_i]$
The problem is now reduced to the length \( S(x) \) of the slices which are obtained by slicing the surface of the body with planes perpendicular to the \( x \)-axis. The pressure is determined by \( P = \frac{F}{A} \), where \( A \) is the surface area of the surface.

**Note 10** If one asks to calculate the force and pressure of a tank at the one vertical end of the tank, one chooses the tank as the body and proceeds in the same way as the previous case (problem 2).

### 9.2.3 Salt in a tank

**Problem 3** A tank contains \( A_1 \) kg of salt dissolved in \( A_2 \) L of water. \( N \) brines enter the tank at the rate of \( r_i \) L/min, \( (i = 1..N) \), containing \( c_i \) kg, \( (i = 1..N) \), of salt per liter of water (salt-concentration). The solution is kept thoroughly mixed and drains from the tank at the rate \( \sum_{i=1}^{N} r_i \). How much salt remains in the tank after \( T \) minutes?

**Solution:** \( y(t) \) denotes the amount salt dissolved in water at the time \( t \).

- rate: \( (r_N) \) L/min
  - conc.: \( c_N \) kg/L
- etc.
- rate: \( (r_1) \) L/min
  - conc.: \( c_1 \) kg/L

- use the following general idea:
  \[
  \frac{dy}{dt} = \text{(rate in)} - \text{(rate out)}
  \]

- \( \text{(rate in)} := \sum_{i=1}^{N} c_i r_i \)

- \( \text{(rate out)} := \left( \sum_{i=1}^{N} r_i \right) \frac{y(t)}{A_2} \)

- initial condition:
  \[
  A_1 = y(0)
  \]
It remains to solve the following initial-value problem:

\[
\frac{dy}{dt} = \left( \sum_{i=1}^{N} C_i \theta_i \right) - y(t) \frac{\sum_{i=1}^{N} r_i}{A_2}
\]

\[y(0) = A_1,\]

where all constants are given or can be determined by the given data.