Assignment # 1  
(Due date: Friday, Sept. 13, 2002)

Problem 1.  
(a) Exhibit a function $f: \mathbb{R} \to \mathbb{R}$ which is continuous precisely at one point, when $\mathbb{R}$ is given the usual topology.

(b) Let $X$ be any set, and define $\mathcal{T} = \{U \subset X \mid X - U \text{ is countable } \} \cup \{X, \emptyset\}$. Show that $\mathcal{T}$ is a topology on $X$.

Problem 2.  
Recall that a map $f: X \to Y$ between two topological spaces is open if whenever $U \subset X$ is open, then $f(U) \subset Y$ is open in $Y$. Consider $X = \mathbb{R}^2$ with the topology of the lexicographic order and $\mathbb{R}$ with the Euclidean metric topology, and let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be the projection on the first and second factors, respectively. More precisely, $f(x, y) = x$ and $g(x, y) = y$. Show that $f$ is not an open map, and that $g$ is an open map.

Problem 3.  
Let $(X, d)$ be a metric space.

(a) Show that $\rho(x, y) := \frac{d(x, y)}{1 + d(x, y)}$ is a metric in $X$ and that $\rho$ and $d$ are equivalent metrics, in other words, they induce the same topology on $X$.

(b) Define $D(x, y) = \min\{1, d(x, y)\}$. Show that $D$ defines a metric in $X$, which is also equivalent to $d$.

Problem 4.  
Recall that a map $f: X \to Y$ between two topological spaces is open if whenever $U \subset X$ is open, then $f(U) \subset Y$ is open in $Y$.

Consider $X = \mathbb{R}^2$ with the topology of the lexicographic order and $\mathbb{R}$ with the Euclidean metric topology, and let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be the projection on the first and second factors, respectively. More precisely, $f(x, y) = x$ and $g(x, y) = y$. Show that $f$ is not an open map, and that $g$ is an open map.

Problem 5.  
Recall that a set $A$ is a topological space $X$ is said to be nowhere dense in $X$ if $\overline{A}$ contains no nonempty open sets.

(a) Show that in any space $X$, the boundary of an open set is closed and nowhere dense.

(b) Conversely, show that every closed nowhere dense set is the boundary of an open set.
Problem 6. An open subset \( U \) in a topological space is said to be **regularly open** if \( U \) is the interior of its closure. A closed set is **regularly closed** if it is the closure of its interior. Show:

a. The complement of a regularly open set is regularly closed and vice-versa.

b. There are open sets in \( \mathbb{R} \) (with the Euclidean topology) which are not regularly open.

c. If \( A \) is any subset of a topological space then \( \text{int}(\overline{A}) \) (the interior of the closure of \( A \)) is regularly open.


Problem 10. Let \( L \subset \mathbb{R}^2 \) be the subset consisting of all points in the plane with coordinates in \( \mathbb{Z} \), i.e., \( L = \{(m, n) \in \mathbb{R}^2 \mid m, n \in \mathbb{Z}\} \). Assume the following FACT: “If a polynomial \( f(x, y) \) in two variables, with real coefficients, has a zero at all points of \( L \), then \( f(x, y) \) is the zero polynomial.”

Now, let \( \mathbb{R}^2_Z \) denote the plane with the Zariski topology (introduced in classroom).

a. Show that \( \overline{L} = \mathbb{R}^2_Z \), in other words, \( L \) is dense in \( \mathbb{R}^2_Z \).

b. Let \( f(x, y) \) and \( g(x, y) \) be two polynomials as above, and define a function \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) by \( F(x, y) = (f(x, y), g(x, y)) \). Show that \( F \) continuous in the Zariski topology.

c. Let \( G : \mathbb{R}^2_Z \to \mathbb{R} \) be defined by \( G(x, y) = \sin (2\pi xy) \). Is \( G \) continuous in the Zariski topology? Explain.