Problem 1. Let $X$ be a paracompact space that contains a dense Lindelöf subspace $D$. Prove that $X$ is also Lindelöf.

Problem 2. Define a relation on $\mathbb{R}^2$ by $(x_1, y_1) \sim (x_2, y_2)$ provided $x_1 + y_1^2 = x_2 + y_2^2$. Show that $\sim$ is an equivalence relation on $\mathbb{R}^2$ and describe the resulting identification space.

Problem 3. Show that the subset of $S^n$ defined by the inequality $x_2^1 + \cdots x_k^2 \leq x_{k+1}^2 + \cdots x_{n+1}^2$ is homeomorphic to $D^k \times S^{n-k}$. (Here, $D^k$ is the closed unit ball in $\mathbb{R}^k$.)

Problem 4. Define functions from $\mathbb{R}$ to $\mathbb{R}^2$ as follows: $f(t) = (t, t^3)$, $g(t) = (t^2, t^3)$, $h(t) = (t^3, t^5)$, $k(t) = (\cos t, \sin t)$.

a: Show that $f$ is a smooth embedding of $\mathbb{R}$ into $\mathbb{R}^2$.

b: Show that $h$ is a topological embedding of $\mathbb{R}$ into $\mathbb{R}^2$ but it is not an immersion.

c: Is $g$ an immersion? An embedding?

d: Is $k$ an immersion? An embedding?

Problem 5. Let $i : M \hookrightarrow \mathbb{R}^n$ be a smooth embedding of an $m$-manifold $M$ as a closed submanifold of $\mathbb{R}^n$, and let $f : M \to \mathbb{R}$ be a smooth function.

a: Show that for each point $p \in M$ one can find an open neighborhood $U$ of $i(p)$ in $\mathbb{R}^n$ and a smooth function $F_U : U \to \mathbb{R}$ whose restriction to $U \cap M$ coincides with the restriction of $f$ to $U \cap M$.

b: Use the previous item to prove that one can find a smooth function $F : \mathbb{R}^n \to \mathbb{R}$ such that $f = F \circ i$, i.e., the restriction of $F$ to $M$ is $f$.

Problem 6. a: Given $p \in M$, explain why $T_p M$ can be identified to a linear subspace of $T_p \mathbb{R}^n \equiv \mathbb{R}^n$ under $i_*$. 

b: Let $f : U \to \mathbb{R}$ be a smooth real valued function defined on an open neighborhood $U \subset \mathbb{R}^n$ of $p \in M$, which is constant on $M$. Show that the gradient $\nabla f$ is perpendicular to $T_p M$.

(REMARK: In the last statement we are identifying $M$ with its image $i(M) \subset \mathbb{R}^r$ under $i$. Recall that the gradient of $f$ at $x \in \mathbb{R}^n$ is the vector $\nabla f(x) = (\frac{\partial f}{\partial x_1} |_x, \ldots, \frac{\partial f}{\partial x_n} |_x)$.)
Problem 7.

a: Let \( p, q \in B \) be distinct points in an open ball \( B \subset \mathbb{R}^n \). Given vectors \( u, v \in \mathbb{R}^n \), show that one can find a smooth curve \( \gamma : (-\epsilon, 1+\epsilon) \to B \) such that \( \gamma(0) = p, \gamma(1) = q \) and \( \gamma'(0) = u, \gamma'(1) = v \).

b: Let \( M^m \) be a smooth manifold of dimension \( m \). Explain why the connected components of \( M \) are the same as the path-components.

c: Explain why \( M \) has at most countably many connected components.

d: Suppose that \( M \) is connected, and let \( p, q \) be points in \( M \). Show that there is a smooth curve \( \gamma : [0,1] \to M \) with \( \gamma(0) = p \) and \( \gamma(1) = q \).

Problem 8. Let \( f : M \to N \) be a continuous map from a space \( M \) into a connected smooth \( n \)-dimensional manifold \( N \). Suppose that every point \( y \in N \) has a neighborhood \( V \) such that \( f^{-1}(V) \) is a disjoint union \( U_1 \ldots \cup U_k \) of open subsets of \( M \) with the property that the restriction \( f|_{U_i} : U_i \to V \) is a homeomorphism.

a: Show that \( f \) is both an open map and an identification map.

b: Show that the number \( k \) in the disjoint union above is the same for every point \( y \in N \).

c: Show that \( f \) induces a structure of smooth manifold of dimension \( n \) on \( M \) so that \( f \) is a smooth map and a local diffeomorphism. (In particular, \( f \) is a submersion and an immersion, but \( f \) is not necessarily an embedding.)

Problem 9. Let \( F_{k,n} \subset \mathbb{R}^n \times \cdots \times \mathbb{R}^n \) denote the subset consisting of \( k \)-tuples \( (v_1, \ldots, v_k) \) of orthonormal vectors (i.e. \( |v_i| = 1 \) for all \( i \) and \( v_i \cdot v_j = 0 \) if \( i \neq j \)).

a: Show that \( F_{k,n} \) is a compact smooth submanifold of \( \mathbb{R}^n \times \cdots \times \mathbb{R}^n \), and determine its dimension.

b: Exhibit a surjective smooth map \( O(n) \to F_{k,n} \), where \( O(n) \) is the orthogonal group, and explain why this is an identification map (a quotient map).

Problem 10. Let \( SL_n(\mathbb{R}) \) denote the set of \( n \times n \) real matrices with determinant 1.

a: Show that \( SL_n(\mathbb{R}) \) is a smooth manifold and compute its dimension. Explain why \( SL_n(\mathbb{R}) \) is a Lie Group.

b: Since \( SL_n(\mathbb{R}) \) is a submanifold of \( M_{n \times n}(\mathbb{R}) \), one can identify the tangent space \( T_A SL_n(\mathbb{R}) \) with a linear subspace of \( M_{n \times n}(\mathbb{R}) \), cf. Problem 2. Prove that \( T_I SL_n(\mathbb{R}) \) is the set of matrices with trace equal to 0.