Study the material in pages 211, 212 and 213 of the textbook.

Final Exam
(Due date: Thursday, Dec. 16, 2004)

Problem 1. Use the cup product to compute the map in cohomology
\( f^*: H^*(\mathbb{C}P^n; \mathbb{Z}) \rightarrow H^*(\mathbb{C}P^n; \mathbb{Z}) \) induced by the map \( f: \mathbb{C}P^n \rightarrow \mathbb{C}P^n \)
described in homogeneous coordinates by \( f([z_0 : z_1 : \cdots : z_n]) = [z_0^d : z_1^d : \cdots : z_n^d] \), where \( d \) is a positive integer.

HINT: Do first the case \( n = 1 \) and use the computation of the cohomology ring of \( \mathbb{C}P^n \).

Problem 2. Let \( X \) and \( Y \) be based spaces.

a: Describe the cohomology ring of \( X \vee Y \) (with coefficients in a ring \( R \)) in terms of the cohomology rings of \( X \) and \( Y \).

b: Show that \( \mathbb{R}P^3 \) and \( \mathbb{R}P^2 \vee S^3 \) have the same cohomology groups but not the same cohomology rings. Can these two spaces be homotopy equivalent?

Problem 3. Show that the ring \( H^*(\mathbb{R}P^\infty; \mathbb{Z}/2k) \) is isomorphic to
\[
\mathbb{Z}/m[\alpha, \beta]/\langle 2\alpha, 2\beta, \alpha^2 - k\beta \rangle,
\]
if \( k \geq 1 \), where \( \alpha \in H^1(\mathbb{R}P^\infty; \mathbb{Z}/2k) \) and \( \beta \in H^2(\mathbb{R}P^\infty; \mathbb{Z}/2k) \).

HINT: Use the map of coefficients \( \mathbb{Z}/2k \rightarrow \mathbb{Z}/2 \), together with the proof of Theorem 3.12.

Problem 4. Show that if \( f: S^n \rightarrow S^n \) is a map of degree \( d \), then the map \( f^*: H^n(S^n; G) \rightarrow H^n(S^n; G) \) is multiplication by \( d \), for any abelian group \( G \).

Problem 5. Study Example 2.43 in your textbook and solve Problem 10, p. 205 (Section 3.1).