1. Show that the product $AB$ and $BA$ have the same spectrum, i.e. $\sigma(AB) = \sigma(BA)$, for any $A, B \in \mathbb{C}^{n\times n}$. Is this statement true if $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$ with $n \neq m$? Prove it or give a counterexample.

2. For $A \in \mathbb{C}^{n\times n}$ define

$$r(A) = \sup_{0 \neq x \in \mathbb{C}^n} \frac{|x^H A x|}{x^H x}$$

(called numerical radius of the matrix $A$).

(a) Show that $r(\cdot)$ has the following properties (common for a norm in $\mathbb{C}^{n\times n}$): $r(A) \geq 0$ and $r(A) = 0$ only if $A = 0$, $r(A + B) \leq r(A) + r(B)$, $r(\lambda A) = |\lambda| r(A)$;

(b) Show that $r(A) \geq \rho(A)$ and $r(A) = \rho(A)$ if $A$ is normal (here $\rho(A)$ is the spectral radius of $A$);

(c) (10 bonus points) Show that $r(A) \leq ||A||_2 \leq 2r(A)$.

3. Let $A$ and $B$ be $n \times n$ upper triangular matrices. Show that $AB$ is a upper triangular matrix and $A^{-1}$ is also a upper triangular matrix if $A$ is non-singular.

4. Let $A \in \mathbb{C}^{n\times n}$ and $P(z) = \sum_{i=0}^{k} a_i z^i$ be a polynomial of degree $k$ of the complex variable $z$. Prove that

$$\sigma(P(A)) = \{ \mu : \mu = P(\lambda), \lambda \in \sigma(A) \}.$$