Consider the linear algebraic system $Au = b$ obtained after the discretization of the Laplace equation for $u(x, y)$ with homogeneous Dirichlet boundary conditions in a unit square:

$$-\Delta u = 1, \quad \text{in } \Omega = (0, 1) \times (0, 1), \quad u = 0, \quad \text{on } \partial \Omega.$$ 

Consider the mesh $(x_i, y_j) = (hi, jh), \ i, j = 0, 1, \ldots, n + 1, \ h = 1/(n + 1)$ and following two approximations: (1) 5-point finite difference formula; (2) 9-point finite difference of higher order. For each of the above two linear systems written as $AU = b$, where the unknowns on $\partial \Omega$ are set to zero and excluded from the system, find $0 < \lambda < \Lambda$ such that $\lambda(DU, U) \leq (AU, U) \leq \Lambda(DU, U)$, for all $U \in \mathbb{R}^{m^2}$ (here $D$ is the diagonal of $A$) that are necessary for Chebyshev acceleration method.

(1) Implement the non-stationary Richardson iteration method (or acceleration by Chebyshev polynomials by **two-term recurrence relation**):

$$DU^{k+1} = ((D - \tau_{k+1}A)U^k + \tau_{k+1}b, \ k = 0, \ldots, m - 1, \ U^0 = (1, \ldots, 1)^T.$$ 

The iteration parameters are chosen in a cyclic way: for a given $m$ (below we use just $m = 8$ and $m = 32$) compute

$$\tau_0 = \frac{2}{\lambda + \Lambda}, \ \rho_0 = \frac{\kappa - 1}{\kappa + 1}, \ \kappa = \frac{\Lambda}{\lambda}, \ \tau_k = \frac{\tau_0}{1 - \rho_0 \mu_k}, \ \mu_k \in \Pi_m, \ k = 1, \ldots, m;$$

here

$$\Pi_m = \{-cos\beta_i, \ \beta_i = \frac{\pi}{2m} \theta^{(m)}_i, \ i = 1, 2, \ldots, m\}.$$ 

The following two sets are proposed for $m = 8$ and for $m = 32$:

$$\theta^{(8)} = \{1, 3, 5, 7, 9, 11, 13, 15\} \ (\text{unstable}), \ \theta^{(8)}_{\text{stable}} = \{1, 15, 7, 9, 3, 13, 5, 11\};$$

$$\theta^{(32)} = \{1, 3, 5, \ldots, 65\} \ (\text{unstable})$$

$$\theta^{(32)}_{\text{stable}} = \{1, 63, 31, 33, 15, 49, 17, 47, 7, 57, 25, 39, 9, 55, 23, 41, \ldots, 3, 61, 29, 35, 13, 51, 19, 45, 5, 59, 27, 37, 11, 53, 21, 43\}.$$ 

(2) Implement also the **three-term recurrence relation** of Chebyshev acceleration:

(a) for $\alpha_1 = 2$, compute $U^1 = U^0 + \tau_0D^{-1}(b - AU^0)$;

(b) for $j = 1, 2, \ldots$, compute

$$\alpha_{j+1} = 4/(4 - \rho_0^2 \alpha_j), \quad U^{j+1} = \alpha_{j+1}U^j + (1 - \alpha_{j+1})U^{j-1} + \alpha_{j+1}\tau_0D^{-1}(b - AU^j).$$ 

In tables report the number of iterations that are necessary to obtain $||r^m||_2/||r^0||_2 < 10^{-7}$, for $n = 10, 20, 40, 80$. Here $r^m = b - AU^m$ is the residual of the $m$-th iterate. Comments on the number of iterations compared with the results of the Programming Assignment # 2.

Remark 1. First find estimates for the parameters $\lambda$, and $\Lambda$ in order to find an approximation to the iteration parameters $\tau_k$.

Remark 2. In the case of $\theta^{(32)}$ you may get an overflow in your computations. However, for $\theta^{(32)}_{\text{stable}}$ you should have no computational complications.