Each easy problem is 10 pts while the medium ones are 20 pts each. Below \((x, y)\) denotes the Euclidean inner product in \(\mathbb{C}^n\).

**Problem 1** (easy). Let \(A \in \mathbb{C}^{n \times n}\) be a Hermitian positive definite matrix. Show that the largest element of the matrix is positive and is one of the diagonal elements \(a_{ii}, \; i = 1, ..., n\). Show also that \(\det A > 0\).

**Problem 2** (easy). Let \(A \in \mathbb{C}^{n \times n}\) and \(A^H = -A\). Show that: (a) the eigenvalues of \(A\) are purely imaginary numbers; (b) \(I - A\) is non-singular; (c) \(C = (I - A)^{-1}(I + A)\) is unitary.

**Problem 3** (easy). Let \(A = B + iC\), where \(B\) and \(C\) have real entries and \(M = \begin{bmatrix} B & -C \\ C & B \end{bmatrix}\). Show that \(A\) is Hermitian if and only if \(M\) is symmetric. For \(A\) Hermitian, express the eigenvalues and eigenvectors of \(M\) in terms of those of \(A\).

**Problem 4** (easy). Let \(A\) be a \(2 \times 2\) block matrix. Prove that \(A\) and \(2D - A\) (here \(D\) is the block-diagonal of \(A\)) have the same eigenvalues, i.e. \(\sigma(A) = \sigma(2D - A)\).

**Problem 5** (medium) Let \(A \in \mathbb{R}^{n \times n}\) and \(B \in \mathbb{R}^{n \times n}\) be s.p.d. matrices and assume that
\[
A = B - E - E^T, \quad G_{\omega}^{SOR} = (B - \omega E)^{-1}((1 - \omega)B + \omega E^T),
\]
where \(E\) is an arbitrary matrix such that \(B - \omega E\) is non-singular for all \(0 < \omega < 2\). Prove that “SOR iteration” converges, i.e. \(\rho(G_{\omega}^{SOR}) < 1\), and show also that \(\rho(G_{\omega}^{SOR}) \leq ||G_{\omega}^{SOR}||_A < 1\).

**Problem 6** (medium) We consider the system of linear equations \(Ax = b\) where \(A\) is Hermitian and positive definite matrix in \(\mathbb{C}^{n \times n}\) and take \(x^0\) be an arbitrary vector in \(\mathbb{C}^n\).

Consider the following product iteration:
\[
x^{m+1} = (I - \tau_1 A)x^m + \tau_1 b, \quad x^{m+2} = (I - \tau_2 A)x^{m+1} + \tau_2 b \quad \text{for} \quad m = 0, 2, \ldots.
\]
Assume that \(0 < \lambda(x, x) \leq (Ax, x) \leq \Lambda(x, x)\) for all \(x \in \mathbb{C}^n\) for some known constants \(\lambda\) and \(\Lambda\).

(a) Find the iteration parameters \(\tau_1\) and \(\tau_2\) (in terms of \(\lambda\) and \(\Lambda\)) so that the convergence is the fastest possible;

(b) Compute the parameters and the number of iterations for achieving relative accuracy \(\epsilon\) in the Euclidean norm for the matrix \(A\) with entries \((-1, 2, -1)\) on the lower co-diagonal, diagonal and upper co-diagonal, correspondingly.

**Problem 7** (medium). In the system \(Ax = b\) the matrix \(A \in \mathbb{C}^{n \times n}\) is Hermitian and non-negative definite, i.e. \((Ax, x) \geq 0\) for all \(x \in \mathbb{C}^n\). Suppose that the only eigenvector corresponding to the zero eigenvalue is the vector \(1 = (1, 1, ..., 1)^T\) and the right-hand side \(b\) satisfies the condition \((b, 1) = 0\). Show that there is a unique solution of this system that satisfies the additional condition \((x, 1) = 0\). To find this solution we are applying the Jacobi method
\[
x^{(m+1)} = x^{(m)} - \tau(Ax^{(m)} - b).
\]
Specify \(x^{(0)}\) and the iteration parameter \(\tau\) so that the method converges to the solution.