MATH 661-600  
Homework #1  
Finite element method for 1-D problems  
Due, September 12, 2003

Below we use the notations from class and from the textbook: S.C. Brenner and L.R. Scott,  
The Mathematical Theory of Finite Element Methods, Springer-Verlag, 1994. Namely, $V = \{v, v' \in L^2(0, 1): v(0) = 0\}$, and $0 = x_0 < x_1 < \ldots < x_n = 1$ is a mesh in the interval $[0, 1]$. Further, $S \subset V$ is the space of piece-wise linear functions over the mesh.

1. (20 pts) (problem 0.x.7, p. 18 of your textbook) Let $h = \max_{1 \leq i \leq n}(x_i - x_{i-1})$. Then

$$\inf_{v \in S} \|u - v\| \leq Ch^2\|u''\|.$$ 

2. (20 pts) (problem 0.x.8, p. 19 of your textbook) Prove that the problem $-u'' = f(x), x \in (0, 1), u(0) = 0, u'(1) = 0$ has a solution $u \in C^2([0, 1])$ provided $f \in C^0([0, 1])$.

3. (20 pts) Let $a(u, v) = (u', v')$. Prove:

(a) (problem 0.x.9, p. 19 of your textbook) the following coercivity result:

$$\|v\|^2 + \|v'\|^2 \leq C_1 a(v, v), \forall v \in V;$$

(b) (problem 0.x.10, p. 19 of your textbook) the following version of Sobolev's inequality:

$$\|v\|_{\max}^2 \leq C_2 a(v, v), \forall v \in V;$$

Give the value for the constants $C_1$ and $C_2$. For simplicity, restrict the result to $v \in V \cap C^1([0, 1])$.

4. (40 pts) Consider the b.v.p.:

$$-u'' + bu' = f(x), x \in (0, 1), u(0) = 0, u'(1) = 0.$$ 

Here $b$ is a positive constant. Consider the finite element approximation $u_S$ for this problem with linear elements. Construct the corresponding Green’s function and provide an estimate for the error in maximum-norm:

(1) First estimate $|u(x_i) - u_S(x_i)|$ for $i = 1, \ldots, n$;

(2) Next, bound $\max_{x \in [0, 1]} |u(x) - u_S(x)|$ (the error in max-norm).