Chapter 1

Some Basic Background

In this chapter we want to recall some important basic results from Functional Analysis most of which were already covered in the Real Analysis course Math607,608 and can be found in the textbooks [Fol] and [Roy].

1.1 Normed Linear Spaces, Banach Spaces

All our vectors spaces will be vector spaces over the real field $\mathbb{R}$ or the complex $\mathbb{C}$. In the case that the field is undetermined we denote it by $K$.

**Definition 1.1.1.** [Normed linear spaces]

Let $X$ be a vector space over $K$, with $K = \mathbb{R}$ or $K = \mathbb{C}$. A *semi norm* on $X$ is a function $\| \cdot \| : X \to [0, \infty)$ satisfying the following properties for all $x, y \in X$ and $\lambda \in K$

1. $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality) and
2. $\|\lambda x\| = |\lambda| \cdot \|x\|$ (homogeneity),

and we call a semi norm $\| \cdot \|$ a *norm* if it also satisfies

3. $\|x\| = 0 \iff x = 0$, for all $x \in X$.

In that case we call $(X, \| \cdot \|)$, or simply $X$, a *normed space*. Sometimes we might denote the norm on $X$ by $\| \cdot \|_X$ to distinguish it from some other norm $\| \cdot \|_Y$ defined on some other space $Y$.

For a normed space $(X, \| \cdot \|)$ the sets

$$B_X = \{x \in X : \|x\| \leq 1\} \text{ and } S_X = \{x \in X : \|x\| = 1\}$$
are called the unit ball and the unit sphere of $X$, respectively.

Note that a norm $\| \cdot \|$ on a vector space defines a metric $d(\cdot, \cdot)$ by
\[
d(x, y) = \|x - y\|, \quad x, y \in X,
\]
and this metric defines a topology on $X$, also called the strong topology.

**Definition 1.1.2.** [Banach Spaces]
A normed space which is complete, i.e. in which every Cauchy sequence converges, is called a Banach space.

To verify that a certain norm defines a complete space it is enough, and sometimes easier to to verify that absolutely converging series are converging:

**Proposition 1.1.3.** Assume that $X$ is a normed linear space so that for all sequences $(x_n) \subset X$ for which $\sum \|x\|_n < \infty$, the series $\sum x_n$ converges (i.e. $\lim_{n \to \infty} \sum_{j=1}^n x_j$ exists in $X$).

Then $X$ is complete.

**Proposition 1.1.4.** [Completion of normed spaces]
If $X$ is a normed space, then there is a Banach space $\tilde{X}$ so that:

There is an isometric embedding $I$ from $X$ into $\tilde{X}$, meaning that $I : X \to \tilde{X}$ is linear and $\|I(x)\| = \|x\|$, for $x \in X$, so that the image of $X$ under $I$ is dense in $\tilde{X}$.

Moreover $\tilde{X}$ is unique up to isometries, meaning that whenever $Y$ is a Banach space for which there is an isometric embedding $J : X \to Y$, with dense image, then there is an isometry $\tilde{J} : \tilde{X} \to Y$ (i.e. a linear bijection between $\tilde{X}$ and $Y$ for which $\|\tilde{J}(\tilde{x})\| = \|\tilde{x}\|$ for all $\tilde{x} \in \tilde{X}$), so that $\tilde{J} \circ I(x) = J(x)$ for all $x \in X$.

The space $\tilde{X}$ is called a completion of $X$.

Let us recall some examples of Banach spaces.

**Examples 1.1.5.** Let $(\Omega, \Sigma, \mu)$, and let $1 \leq p < \infty$ a measure space then
\[
\mathcal{L}_p(\mu) := \left\{ f : \Omega \to \mathbb{K} \ \text{mble} : \int_\omega |f|^p d\mu(x) < \infty \right\}.
\]

For $p = \infty$ we put
\[
\mathcal{L}_\infty(\mu) := \left\{ f : \Omega \to \mathbb{K} \ \text{mble} : \exists C \ \mu(\{\omega \in \Omega : |f(\omega)| > C\}) = 0 \right\}.
\]
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Then $\mathcal{L}_p(\mu)$ is a vector space, and the map

$$
\| \cdot \|_p : \mathcal{L}_p(\mu) \to \mathbb{R}, \quad f \mapsto \left( \int_{\Omega} |f(\omega)|^p d\mu(\omega) \right)^{1/p},
$$

if $1 \leq p < \infty$, and

$$
\| \cdot \|_\infty : \mathcal{L}_\infty(\mu) \to \mathbb{R}, \quad f \mapsto \inf \{ C > 0 : \mu(\{ \omega \in \Omega : |f(\omega)| > C \}) = 0 \},
$$

if $p = \infty$, is a seminorm on $\mathcal{L}_p(\mu)$.

For $f, g \in \mathcal{L}_p(\mu)$ define the equivalence relation by

$$
f \sim g : \iff f(\omega) = g(\omega) \text{ for } \mu\text{-almost all } \omega \in \Omega.
$$

Define $L_p(\mu)$ to be the quotient space $\mathcal{L}_p(\mu)/\sim$. Then $\| \cdot \|_p$ is well defined and a norm on $L_p(\mu)$, and turns $L_p(\mu)$ into a Banach space. Although, strictly speaking, elements of $L_p(\mu)$ are not functions but equivalence classes of functions, we treat the elements of $L_p(\mu)$ as functions, by picking a representative out of each equivalence class.

If $A \subset \mathbb{R}$, or $A \subset \mathbb{R}^d$, $d \in \mathbb{N}$, and $\mu$ is the Lebesgue measure on $A$ we write $L_p(A)$ instead of $L_p(\mu)$. If $\Gamma$ is a set and $\mu$ is the counting measure on $\Gamma$ we write $\ell_p(\Gamma)$ instead of $L_p(\mu)$. Thus

$$
\ell_p(\Gamma) = \left\{ x(\cdot) : \Gamma \to \mathbb{K} : \| x \|_p = \left( \sum_{\gamma \in \Gamma} |x_\gamma|^p \right)^{1/p} < \infty \right\}, \text{ if } 1 \leq p < \infty, \text{ and }
$$

$$
\ell_\infty(\Gamma) = \left\{ x(\cdot) : \Gamma \to \mathbb{K} : \| x \|_\infty = \sup_{\gamma \in \Gamma} |x_\gamma| < \infty \right\}.
$$

If $\Gamma = \mathbb{N}$ we write $\ell_p$ instead of $\ell_p(\mathbb{N})$ and if $\Gamma = \{1, 2 \ldots n\}$, for some $n \in \mathbb{N}$ we write $\ell_p^n$ instead of $\ell_p(\{1, 2 \ldots n\})$.

The set

$$
c_0 = \{(x_n : n \in \mathbb{N}) \subset \mathbb{K} : \lim_{n \to \infty} x_n = 0\}
$$

is a linear closed subspace of $\ell_\infty$, and, thus, also a Banach space (with $\| \cdot \|_\infty$).

More generally, let $S$ be a (topological) Hausdorff space, then

$$
C_b(S) = \{ f : S \to \mathbb{K} \text{ continuous and bounded } \}
$$

is a closed subspace of $\ell_\infty(S)$ and, thus, a Banach space. If $K$ is a compact space we will write $C(K)$ instead of $C_b(K)$. If $S$ is locally compact then

$$
C_0(S) = \{ f : S \to \mathbb{K} \text{ continuous and } \{|f| \geq c\} \text{ is compact for all } c > 0 \}.$$
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is a closed subspace of $C_b(S)$ and, thus a Banach space.

Let $(\Omega, \Sigma)$ be a measurable space and assume first that $\mathbb{K} = \mathbb{R}$. Recall that a finite signed measure on $(\Omega, \Sigma)$ is a map $\mu : \Sigma \to \mathbb{R}$ so that $\mu(\emptyset) = 0$, and so that

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n), \quad \text{whenever } (E_n) \subset \Sigma \text{ is pairwise disjoint}.$$ 

The Jordan Decomposition Theorem says that such a signed measure can be uniquely written as the difference of two positive finite measure $\mu^+$ and $\mu^-$. Letting now

$$\|\mu\|_v = \mu^+(\Omega) + \mu^-(\Omega) = \sup_{A,B \in \Sigma, \text{disjoint}} \mu(A) - \mu(B),$$

Then $\|\cdot\|_v$ is a norm, the *variation norm*, on

$$M(\Sigma) = M_\mathbb{R}(\Sigma) := \{\mu : \Sigma \to \mathbb{R} : \text{signed measure}\},$$

which turns $M(\Sigma)$ into a real Banach space.

If $\mathbb{K} = \mathbb{C}$, we define

$$M(\Sigma) = M_\mathbb{C}(M) = \{\mu + i\nu : \mu, \nu \in M_\mathbb{R}(\Sigma)\},$$

and define for $\mu + i\nu \in M_\mathbb{C}(\Sigma)$

$$\|\mu + i\nu\|_v = \sqrt{\|\mu\|_v^2 + \|\nu\|_v^2}.$$}

Then $M_\mathbb{C}(\Sigma)$ is a complex Banach space.

Assume $S$ is a topological space and $\mathcal{B}_S$ is the sigma-algebra of Borel sets, i.e. the $\sigma$-algebra generated by the open subsets of $S$. We call a (positive) measure on $\mathcal{B}$ a Radon measure if it is outer regular, i.e.

$$\mu(A) = \inf\{\mu(U) : U \subset S \text{ open and } A \subset U\} \text{ for all } A \in \mathcal{B}_S,$$

inner regular, i.e.

$$\mu(A) = \sup\{\mu(C) : C \subset S \text{ compact and } C \subset A\} \text{ for all } A \in \mathcal{B}_S,$$

and if it is finite on all compact subsets of $S$. A signed Radon measure is then a difference of two finite positive Radon measure. We denote the set of all signed Radon measures by $M(S)$. Then $M(S)$ is a closed linear subspace of $M(\mathcal{B}_S)$. 
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Remark. \( M(B_{\mathbb{R}}) = M(\mathbb{R}) \).

There are many ways to combine Banach spaces to new spaces.

**Proposition 1.1.6.** [Complemented sums of Banach spaces]

If \( X_i \) is a Banach space for all \( i \in I \), \( I \) some index set, and let \( 1 \leq p \leq \infty \)

\[
(\oplus_{i \in I} X_i)_\ell^p := \left\{ (x_i)_{i \in I} : x_i \in X_i, \text{ for } i \in I, \text{ and } (\|x_i\| : i \in I) \in \ell^p(I) \right\}.
\]

We put for \( x \in (\oplus_{i \in I} X_i)_\ell^p \)

\[
\|x\|_p := \left\| \left(\|x_i\| : i \in I\right) \right\|_p = \begin{cases} \left(\sum_{i \in I} \|x_i\|^{p}_{X_i}\right)^{1/p} & \text{if } 1 \leq p < \infty, \\
\sup_{i \in I} \|x_i\|_{X_i} & \text{if } p = \infty. 
\end{cases}
\]

Then \( \| \cdot \| \) is a norm on \( (\oplus_{i \in I} X_i)_\ell^p \) and \( (\oplus_{i \in I} X_i)_\ell^p \) is a Banach space. We call \( (\oplus_{i \in I} X_i)_\ell^p \) the \( \ell^p \) sum of the \( X_i, i \in I \).

Moreover,

\[
(\oplus_{i \in I} X_i)_{\ell^\infty}_0 := \left\{ (x_i)_{i \in I} (\oplus_{i \in I} X_i)_{\ell^\infty} : \forall c > 0 \ \{ i \in I : \|x_i\| \geq c \} \text{ is finite} \right\}
\]

is a closed linear subspace of \( (\oplus_{i \in I} X_i)_{\ell^\infty} \) and, and, thus, also a Banach space.

If all the spaces \( X_i \) are the same spaces in Proposition 1.1.6, say \( X_i = X \), for \( i \in I \) we write \( \ell^p(I, X) \), and \( c_0(I, X) \), instead of \( (\oplus_{i \in I} X_i)_{\ell^p} \) or \( (\oplus_{i \in I} X_i)_{\ell^\infty}_0 \), respectively. We write \( \ell^p(X) \), and \( c_0(X) \) instead of \( \ell^p(\mathbb{N}, X) \) and \( c_0(\mathbb{N}, X) \), respectively, and \( \ell^p_n(X) \), instead of \( \ell^p(\{1, 2, \ldots, n\}, X) \), for \( n \in \mathbb{N} \).

**Exercises:**

1. Prove Proposition 1.1.4.

2. Let \( 1 \leq p \leq \infty \), and let \( X_i \) be a Banach space for each \( i \in I \), where \( I \) is some index set. Show that the norm \( \| \cdot \|_p \) introduced in Proposition 1.1.6 on \( (\oplus_{i \in I} X_i)_{\ell^p} \) turns this space into a complete normed space (only show completeness).

3. Let \( f \in L^p[0, 1] \) for some \( p > 1 \). Show that

\[
\lim_{r \to 1^+} \|f\|_r = \|f\|_1.
\]
1.2 Operators on Banach spaces, dual spaces

If $X$ and $Y$ are two normed linear spaces, then for a linear map (we also say linear operator) $T : X \to Y$ the following are equivalent

a) $T$ is continuous,

b) $T$ is continuous at 0,

c) $T$ is bounded, i.e. $\|T\| = \sup_{x \in B_X} \|T(x)\| < \infty$.

In this case $\| \cdot \|$ is a norm on

\[ L(X, Y) = \{ T : X \to Y \text{ linear and bounded} \} \]

which turns $L(X, Y)$ into a Banach space if $Y$ is a Banach space, and we observe that

\[ \|T(x)\| \leq \|T\| \cdot \|x\| \text{ for all } T \in L(X, Y) \text{ and } x \in X. \]

We call a bounded linear operator $T : X \to Y$ an \textit{isomorphic embedding} if there is a number $0 < c$ so that $c\|x\| \leq \|T(x)\|$. This is equivalent to saying that the image $T(X)$ of $T$ is a closed subspace of $Y$ and $T$ has an inverse $T^{-1} : T(X) \to Y$ which is also bounded.

An isomorphic embedding which is onto (we say also surjective) is called an \textit{isomorphy} between $X$ and $Y$. If $\|T(x)\| = \|x\|$ for all $x \in X$ we call $T$ an \textit{isometric embedding}, and call it an \textit{isometry between $X$ and $Y$} if $T$ is surjective.

If there is an isometry between two spaces $X$ and $Y$ we write $X \simeq Y$. In that case $X$ and $Y$ can be identified for our purposes. If there is an isomorphism $T : X \to Y$ with $\|T\| : \|T^{-1}\| \leq c$, for some number $c \geq 1$ we write $X \sim_C Y$ and we write $X \sim Y$ if there is a $C \geq 1$ so that $X \sim_C Y$.

If $X$ and $Y$ are two Banach spaces which are isomorphic (for example if both spaces are finite dimensional and have the same dimension), we define

\[ d_{BM}(X, Y) = \inf\{\|T\| : \|T^{-1}\| : T : X \to Y, T \text{ isomorphism}\}, \]

and call it the \textit{Banach Mazur distance between $X$ and $Y$}. Note that always $d_{BM}(X, Y) \geq 1$. 
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Remark. If \((X, \| \cdot \|_X)\) is a finite dimensional Banach space over \(\mathbb{K}\), \(\mathbb{K} = \mathbb{R}\) or \(\mathbb{K} = \mathbb{C}\), and its dimension is \(n \in \mathbb{N}\) we can after passing to an isometric image, assume that \(X = \mathbb{K}^n\). Indeed, let \(x_1, x_2, \ldots, x_n\) be a basis of \(X\), and consider on \(\mathbb{K}^n\) the norm given by:

\[
\| (a_1, a_2, \ldots, a_n) \| = \left\| \sum_{j=1}^{n} a_j x_j \right\|_X, \quad \text{for} \ (a_1, a_2, \ldots, a_n) \in \mathbb{K}^n.
\]

Then

\[
I : \mathbb{K}^n \to X, \quad (a_1, a_2, \ldots, a_n) \mapsto \sum_{j=1}^{n} a_j x_j,
\]

is an isometry. Therefore we can always assume that \(X = (\mathbb{K}^n, \| \cdot \|_X)\). This means \(B_X\) is a closed and bounded subset of \(\mathbb{K}^n\), which by the Theorem of Bolzano-Weierstraß, means that \(B_X\) is compact. In Theorem 1.5.3 we will deduce the converse and prove that a Banach space \(X\), for which \(B_X\) is compact, must be finite dimensional.

Definition 1.2.1. [Dual space of \(X\)]

If \(Y = \mathbb{K}\) and \(X\) is normed linear space over \(\mathbb{K}\), then we call \(L(X, \mathbb{K})\) the dual space of \(X\) and denote it by \(X^*\).

If \(x^* \in X^*\) we often use \(\langle \cdot, \cdot \rangle\) to denote the action of \(x^*\) on \(X\), i.e. we write \(\langle x^*, x \rangle\) instead of \(x^*(x)\).

Theorem 1.2.2. [Representation of some Dual spaces]

1. Assume that \(1 \leq p < \infty\) and \(1 < q \leq \infty\) with \(\frac{1}{p} + \frac{1}{q} = 1\), and assume that \((\Omega, \Sigma, \mu)\) is a measure space without atoms of infinite measure. Then the following map is well defined an isometry between \(L_p^*(\mu)\) and \(L_q(\mu)\).

\[
\Psi : L_q(\mu) \to L_p^*(\mu), \quad \Psi(g)(f) := \int_{\Omega} f(\xi) g(\xi) \, d\mu(\xi),
\]

for \(g \in L_q(\mu)\), and \(f \in L_p(\mu)\).

is an isometry between \(L_q(\mu)\) and \(L_p^*(\mu)\).

2. Assume that \(S\) is a locally compact Hausdorff space, then the map

\[
\Psi : M(S) \to C_0(S), \quad \Psi(\mu)(f) := \int_S f(\xi) \, d\mu(\xi)
\]

is an isometry between \(M(S)\) and \(C_0^*(S)\).
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for $\mu \in M(S)$ and $f \in C_0(S)$,

is an isometry between $M(S)$ and $C_0^*(S)$.

Remark. If $p = \infty$ and $q = 1$ then the map $\Psi$ in Theorem 1.2.2 part (1) is still an isometric embedding, but in general (i.e. if $L_p(\mu)$ is infinite dimensional) not onto.

Example 1.2.3. $c_0^* \simeq \ell_1$ (by Theorem 1.2.2 part (2)) and $\ell_1^* \simeq \ell_\infty$ (by Theorem 1.2.2 part (1)).

Exercises

1. Let $X$ and $Y$ be normed linear spaces, and $\hat{X}$ and $\hat{Y}$ be completions of $X$ and $Y$, respectively (recall Proposition 1.1.4). Then every $T \in L(X,Y)$ can be extended in a unique way to an element $\hat{T}$ in $L(\hat{X},\hat{Y})$, and $\|\hat{T}\| = \|T\|$.

2. A Banach space $X$ is called strictly convex if for any $x, y \in S_X$, $x \neq y$ $\|x + y\| < 2$.

Prove that $c_0$ and $\ell_1$ are not strictly convex, but that they can be given equivalent norms with which they are strictly convex.

Recall that two norms $\| \cdot \|$ and $\| \cdot \|$ on the same linear space are equivalent if the identity $I : (X, \| \cdot \|) \rightarrow (X, \| \cdot \|)$ is an isomorphism.

3.* A Banach space $X$ is called uniform convex if for for every $\varepsilon > 0$ there is a $\delta$ so that:

If $x, y \in S_X$ with $\|x - y\| > \varepsilon$ then $\|x + y\| < 2 - \delta$.

Prove that $\ell_p$, $1 < p < \infty$ are uniform convex but $\ell_1$ and $c_0$ do not have this property.
1.3 Baire Category Theorem and its consequences

The following result is a fundamental Theorem in Topology and leads to several useful properties of Banach spaces.

**Theorem 1.3.1.** [The Baire Category Theorem]
Assume that $(S, d)$ is a complete metric space. If $(U_n)$ is a sequence of open and dense subsets of $S$ then $\bigcap_{n=1}^{\infty} U_n$ is also dense in $S$.

Often we will use the Baire Category Theorem in the following equivalent restatement.

**Corollary 1.3.2.** If $(C_n)$ is a sequence of closed sets subsets of a complete metric space $(S, d)$ whose union is all of $S$, then there must be an $n \in \mathbb{N}$ so that $C_n^o$, the open interior of $C_n$, is not empty, and thus there is an $x \in C_n$ and an $\varepsilon > 0$ so that $B(x, \varepsilon) = \{z \in S : d(z, x) < \varepsilon\} \subset C_n$.

**Proof.** Assume our conclusion were not true. Let $U_n = S \setminus C_n$, for $n \in \mathbb{N}$. Then $U_n$ is open and dense in $S$. Thus $\bigcap_{n \in \mathbb{N}} U_n$ is also dense, in particular not empty. But this is in contradiction to the assumption that $\bigcup_{n \in \mathbb{N}} C_n = S$. \qed

The following results are important applications of the Baire Category Theorem to Banach spaces.

**Theorem 1.3.3.** [The Open Mapping Theorem]
Let $X$ and $Y$ be Banach spaces and let $T \in L(X, Y)$ be surjective. Then $T$ is also open (the image of every open set in $X$ under $T$ is open in $Y$).

**Corollary 1.3.4.** Let $X$ and $Y$ be Banach spaces and $T \in L(X, Y)$ be a bijection. Then its inverse $T^{-1}$ is also bounded, and thus $T$ is an isomorphism.

**Theorem 1.3.5.** [Closed Graph Theorem]
Let $X$ and $Y$ be Banach spaces and $T : X \to Y$ be linear. If $T$ has a closed graph (i.e $\Gamma(T) = \{(x, T(x)) : x \in X\}$ is closed with respect to the product topology in $X \times Y$), then $T$ is bounded.

Often the Closed Graph Theorem is used in the following way

**Corollary 1.3.6.** Assume that $T : X \to Y$ is a bounded, linear and bijective operator between two Banach spaces $X$ and $Y$. Then $T$ is an isomorphism.
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Theorem 1.3.7. [Uniform Boundedness Principle]
Let $X$ and $Y$ be Banach spaces and let $A \subset L(X,Y)$. If for all $x \in X$ \[ \sup_{T \in A} \|T(x)\| < \infty \] then $A$ is bounded in $L(X,Y)$, i.e.
\[ \sup_{T \in A} \|T\| = \sup_{x \in B_X} \sup_{T \in A} \|T(x)\| < \infty. \]

Proposition 1.3.8. [Quotient spaces]
Assume that $X$ is a Banach space and that $Y \subset X$ is a closed subspace.
Consider the quotient space $X/Y = \{x + Y : x \in X\}$ (with usual addition and multiplication by scalars). For $x \in X$ put $\overline{x} = x + Y \in X/Y$ and define
\[ \|\overline{x}\|_{X/Y} = \inf_{z \in x} \|z\|_X = \inf_{y \in Y} \|x + y\|_X = \text{dist}(x,Y). \]
Then $\|\cdot\|_{X/Y}$ is norm on $X/Y$ which turns $X/Y$ into a Banach space.

Proof. For $x_1, x_2$ in $X$ and $\lambda \in \mathbb{K}$ we compute
\[ \|\overline{x}_1 + \overline{x}_2\|_{X/Y} = \inf_{y \in Y} \|x_1 + x_2 + y\| = \inf_{y_1, y_2 \in Y} \|x_1 + y_1 + x_2 + y_2\| \leq \inf_{y_1, y_2 \in Y} \|x_1 + y_1\| + \|x_2 + y_2\| = \|\overline{x}_1\|_{X/Y} + \|\overline{x}_2\|_{X/Y} \]
and
\[ \|\lambda \overline{x}\|_{X/Y} = \inf_{y \in Y} \|\lambda x + y\| = \inf_{y \in Y} \|\lambda(x_1 + y)\| = |\lambda| \cdot \inf_{y \in Y} \|x_1 + y\| = |\lambda| \cdot \|\overline{x}\|_{X/Y}. \]
Moreover if $\|\overline{x}\|_{X/Y} = 0$ it follows that there is a sequence $\lim_{n \to \infty} \|x - y_n\| = 0$, which implies, since $Y$ is closed that $x = \lim_{n \to \infty} y_n \in Y$ and thus $\overline{x} = 0$ (the zero element in $X/Y$). This proves that $(X/Y, \|\cdot\|_{X/Y})$ is a normed linear space. In order to show that $X/Y$ is complete let $x_n \in X$ with $\sum_{n \in \mathbb{N}} \|\overline{x}_n\|_{X/Y} < \infty$. It follows that there are $y_n \in Y, n \in \mathbb{N}$, so that
\[ \sum_{n=1}^{\infty} \|x + y_n\|_X < \infty, \]
and thus, since $X$ is a Banach space

$$x = \sum_{n=1}^{\infty} (x_n + y_n),$$

exists in $X$ and we observe that

$$\|\bar{x} - \sum_{j=1}^{n} \bar{x}_j\| \leq \|x - \sum_{j=1}^{n} (x_j + y_j)\| \leq \sum_{j=n+1}^{\infty} \|x_j + y_j\| \to_{n \to \infty} 0,$$

which verifies that $X/Y$ is complete.

From Corollary 1.3.4 we deduce

**Corollary 1.3.9.** If $X$ and $Y$ are two Banach spaces and $T : X \to Y$ is a linear, bounded and surjective operator, it follows that $X/N(T)$ and $Y$ are isomorphic, where $N(T)$ is the null space of $T$.

**Proof.** Since $T$ is continuous $N(T)$ is a closed subspace of $X$. We put

$$\bar{T} : X/N(T) \to Y, \quad x + N(T) \mapsto T(x).$$

Then $\bar{T}$ is well defined, linear, and bijective (linear Algebra), moreover, for $x \in X$

$$\|\bar{T}(x+N(T))\| = \inf_{z \in N(T)} \|T(x+z)\| \leq \|T\| \inf_{z \in N(T)} \|x+z\| = \|T\|\|x+N(T)\|_{X/N(T)}.$$

Thus, $\bar{T}$ is bounded and our claim follows from Corollary 1.3.4.

**Proposition 1.3.10.** For a bounded linear operator $T : X \to Y$ between two Banach spaces $X$ and $Y$ the following statements are equivalent:

1. The range $T(X)$ is closed.
2. The operator $\bar{T} : X/N(T) \to Y, \bar{x} \mapsto T(x)$ is an isomorphic embedding,
3. There is a number $C > 0$ so that $\text{dist}(x, N(T)) = \inf_{y \in N} \|x - y\| \leq C\|T(x)\|.$
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Exercises:

1. Prove Proposition 1.3.10.

2. Assume $X$ and $Y$ are Banach spaces and that $(T_n) \subset L(X, Y)$ is a sequence, so that $T(x) = \lim_{n \to \infty} T_n(x)$ exists for every $x \in X$.
   Show that $T \in L(X, Y)$.

3. Let $X$ be a Banach space and $Y$ be a closed subspace. Prove that $X$ is separable if and only if $Y$ and $X/Y$ are separable.