10.2: SERIES

A series is a sum of sequence:

\[ \sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots + a_n + \ldots \]

For a given sequence \({a_k}_{k=1}^{\infty}\) define the following:

The \(s_n\)'s are called partial sums and they form a sequence \({s_n}_{n=1}^{\infty}\).

We want to consider the limit of \({s_n}_{n=1}^{\infty}\):

If \({s_n}_{n=1}^{\infty}\) is convergent and \(\lim_{n \to \infty} s_n = s\) exists as a real number, then the series \(\sum_{k=1}^{n} a_k\) is convergent. The number \(s\) is called the sum of the series.\(^2\)

If \({s_n}_{n=1}^{\infty}\) is divergent then the series \(\sum_{k=1}^{\infty} a_k\) is divergent.

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1. \(k = 1\) for convenience, it can be anything

2. When we write \(\sum_{k=1}^{n} a_k = s\) we mean that by adding sufficiently many terms of the series we can get as close as we like to the number \(s\).
GEOMETRIC SERIES

\[ a + ar + ar^2 + \ldots + ar^{n-1} + \ldots \] \hspace{1cm} (a \neq 0)

Each term is obtained from the preceding one by multiplying it by the common ratio \( r \).

**FACT:** The geometric series is convergent if \(|r| < 1\) and its sum is

\[
\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}.
\]

If \(|r| \geq 1\), the geometric series is divergent.

**EXAMPLE 1.** Determine whether the following series converges or diverges. If it converges, find the sum. If it diverges, explain why.

(a) \( \sum_{n=1}^{\infty} 5 \cdot \left(\frac{2}{7}\right)^n \)

(b) \( \sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}} \)

(c) \( 1 - \frac{3}{2} + \frac{9}{4} - \frac{27}{8} + \ldots \)

(d) \( \sum_{n=1}^{\infty} 4^{n+1} \cdot 9^{2-n} \)
EXAMPLE 2. Write the number $\overline{17}$ as a ratio of integers.

TELESCOPING SUM
Let $b_n$ be a given sequence. Consider the following series:
$$\sum_{n=1}^{\infty} (b_n - b_{n+1})$$

EXAMPLE 3. Determine whether the following series converges or diverges. If it is converges, find the sum. If it is diverges, explain why.

(a) $$\sum_{n=1}^{\infty} \left( \sin \frac{1}{n} - \sin \frac{1}{n+1} \right)$$

(b) $$\sum_{n=1}^{\infty} \ln \frac{n + 1}{n + 2}$$
THEOREM 4. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \to \infty} a_n = 0$.

REMARK 5. The converse is not necessarily true.

THE TEST FOR DIVERGENCE:

If $\lim_{n \to \infty} a_n$ does not exist or if $\lim_{n \to \infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

REMARK 6. If you find that $\lim_{n \to \infty} a_n = 0$ then the Divergence Test fails and thus another test must be applied.

EXAMPLE 7. Use the test for Divergence to determine whether the series diverges.

(a) $\sum_{n=1}^{\infty} \frac{n^2}{3(n+1)(n+2)}$

(b) $\sum_{n=1}^{\infty} \cos \frac{\pi n}{2}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$